2.2 Elementary Functions: Graphs and Transformations

Consider functions \( f(x) = -\frac{1}{4}(x - 3)^2 + 1 \) and \( g(x) = x^2 \).

**Question:** How are these two functions related to each other?

**Answer:**

We know how the graph of \( g(x) = x^2 \) looks like.

**Question:** What can the graph of \( g(x) \) tell us about the graph of \( f(x) \)?

**Answer:**
We will learn a bunch of elementary functions (like \( g(x) = x^2 \)) which may be used as building blocks to construct something more sophisticated (using their composition and applying various transformations). Knowing how elementary functions behave and what effect a particular transformation has allows us to understand behavior of a more complex function we may construct.

A Beginning Library of Elementary Functions

<table>
<thead>
<tr>
<th>Definition (Basic Elementary Functions)</th>
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<tbody>
<tr>
<td>Constant function: ( f(x) = C ) (( C ) is a real number)</td>
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<tr>
<td>Identity function: ( g(x) = x )</td>
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<tr>
<td>Square function: ( h(x) = x^2 )</td>
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<tr>
<td>Cube function: ( m(x) = x^3 )</td>
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<tr>
<td>Square root function: ( n(x) = \sqrt{x} )</td>
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<tr>
<td>Cube root function: ( p(x) = \sqrt[3]{x} )</td>
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<tr>
<td>Absolute value function: ( k(x) =</td>
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</table>

Example 1

(a) If \( h(x) = 2 \), then \( h(14) = \), \( h(-8) = \).
(b) If \( m(x) = x \), then \( m(-3) = \), \( m(5) = \).
(c) If \( p(x) = x^2 \), then \( p(-2) = \), \( p(2) = \).
(d) If \( f(x) = x^3 \), then \( f(-3) = \), \( f(3) = \).
(e) If \( n(x) = \sqrt{x} \), then \( n(16) = \), \( n(-16) = \).
(f) If \( k(x) = \sqrt[3]{x} \), then \( k(-125) = \), \( k(125) = \).
(g) If \( g(x) = |x| \), then \( g(-2) = \), \( g(0) = \).
Question: How does the graph of $f(x) = \frac{5}{2}$ look like?
Answer:

Question: How does the graph of $f(x) = x^3$ look like?
Answer:

Question: How does the graph of $f(x) = |x|$ look like?
Answer:
Notation. \( \mathbb{R} \) (or \( \mathbb{R} \)) denotes ”all real numbers”.

Graphs, Domains and Ranges of Some Elementary Functions

\[ f(x) = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases} \]
Vertical and Horizontal Shifts

**Definition.** If a new function is formed by performing an operation on a given function, then the graph of the new function is called a **transformation** of the graph of the original function.

**Example 2**
Graph the following functions simultaneously in the same coordinate system:

(a) \( y = f(x) = x^2, \ y = g(x) = x^2 + 2, \ y = h(x) = x^2 - 3 \)
(b) \( h = f(x) = x^2, \ y = m(x) = (x + 2)^2, \ y = n(x) = (x - 3)^2 \)
Note that in the previous example we actually had

(a) if $f(x) = x^2$, then $h(x) = x^2 + 2 = f(x) + 2$, $g(x) = x^2 - 3 = f(x) + (-3)$;

(b) if $f(x) = x^2$, then $m(x) = (x + 2)^2 = f(x + 2)$, $n(x) = x^2 - 3 = f(x + (-3))$.

**Vertical Shift**

Comparing the graphs of $y = f(x) + C$ with the graph of $y = f(x)$, we see that the graph of $y = f(x) + C$ can be obtained from the graph of $y = f(x)$ by **vertically translating (shifting)** the graph of the latter upward $C$ units if $C$ is positive and downward $|C|$ units if $C$ is negative.

**Horizontal Shift**

Comparing the graphs of $y = f(x + C)$ with the graph of $y = f(x)$, we see that the graph of $y = f(x + C)$ can be obtained from the graph of $y = f(x)$ by **horizontally translating (shifting)** the graph of the latter $C$ units to the left if $C$ is positive and $|C|$ units to the right if $C$ is negative.
Example 3
Identify the functions which graphs are shown below

(a)

(b)
Reflections, Stretches, and Shrinks

Example 4
Graph the following functions simultaneously in the same coordinate system:

(a) $y = f(x) = |x|$, $y = g(x) = 2|x|$, $y = h(x) = \frac{1}{3}2|x|

(b) $h = f(x) = |x|$, $y = m(x) = −2|x|$, $y = n(x) = −\frac{1}{3}2|x|

(a) (b)

Note that:

(a) if $f(x) = |x|$, then $h(x) = 2|x| = 2f(x)$, $g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$;

(b) if $f(x) = |x|$, then $h(x) = −2|x| = −2f(x)$, $g(x) = −\frac{1}{3}|x| = −\frac{1}{3}f(x)$;
Reflection, Stretch, and Shrink.

Comparing \( y = Cf(x) \) to \( y = f(x) \), we see that the graph of \( y = Cf(x) \) can be obtained from the graph of \( y = f(x) \) by multiplying each ordinate value of the latter by \( C \). The result is a vertical stretch of the graph of \( y = f(x) \) if \( C > 1 \), a vertical shrink of the graph of \( y = f(x) \) if \( 0 < C < 1 \), and a reflection in the \( x \) axis if \( C = -1 \). If \( C \) is a negative number other than -1, then the result is a combination of a reflection in the \( x \) axis and either a vertical stretch or a vertical shrink.

Summary

<table>
<thead>
<tr>
<th>SUMMARY</th>
<th>Graph Transformations</th>
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<tbody>
<tr>
<td><strong>Vertical Translation:</strong></td>
<td></td>
</tr>
<tr>
<td>( y = f(x) + k )</td>
<td>( k &gt; 0 ) Shift graph of ( y = f(x) ) up ( k ) units. ( k &lt; 0 ) Shift graph of ( y = f(x) ) down (</td>
</tr>
<tr>
<td><strong>Horizontal Translation:</strong></td>
<td></td>
</tr>
<tr>
<td>( y = f(x + h) )</td>
<td>( h &gt; 0 ) Shift graph of ( y = f(x) ) left ( h ) units. ( h &lt; 0 ) Shift graph of ( y = f(x) ) right (</td>
</tr>
<tr>
<td><strong>Reflection:</strong></td>
<td></td>
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<tr>
<td>( y = -f(x) )</td>
<td>Reflect the graph of ( y = f(x) ) in the ( x ) axis.</td>
</tr>
<tr>
<td><strong>Vertical Stretch and Shrink:</strong></td>
<td></td>
</tr>
<tr>
<td>( y = Af(x) )</td>
<td>( A &gt; 1 ) Stretch graph of ( y = f(x) ) vertically by multiplying each ordinate value by ( A ). ( 0 &lt; A &lt; 1 ) Shrink graph of ( y = f(x) ) vertically by multiplying each ordinate value by ( A ).</td>
</tr>
</tbody>
</table>
Example 5 (Combing Transformations)
Graph $f(x) = -\frac{1}{4}(x - 3)^2 + 1$
Piecewise-Defined Functions

We have already seen that

\[ f(x) = |x| = \begin{cases} 
-x, & \text{if } x < 0 \\
x, & \text{if } x \geq 0 
\end{cases} \]

The function is defined by different rules for different parts of its domain.

**Definition.** Functions whose definitions involve more than one rule are called **piecewise-defined functions**.

**Question:** How to graph a piecewise-defined functions?

**Answer:** Graph each rule over the appropriate portion of the domain.

**Example 6** (Graphing Piecewise-Defined Functions)

Graph

\[ y = f(x) = \begin{cases} 
(x + 2)^2, & \text{if } x < 0 \\
-2x + 4, & \text{if } 0 \leq x < 3 \\
\sqrt{x - 3}, & \text{if } x \geq 3 
\end{cases} \]