3.1 The Integration Formulas of Newton and Cotes

The integration formulas of Newton and Cotes are obtained if the integrand is replaced by a suitable interpolating polynomial \( P(x) \) and if then \( \int_a^b P(x) \, dx \) is taken as an approximate value for \( \int_a^b f(x) \, dx \). Consider a uniform partition of the closed interval \([a, b]\) given by

\[ x_i = a + \frac{i}{n} \, h, \quad i = 0, \ldots, n, \]

of step length \( h := \frac{b - a}{n}, \ n > 0 \) integer, and let \( P_n \) be the interpolating polynomial of degree \( n \) or less with

\[ P_n(x_i) = f_i := f(x_i) \quad \text{for } i = 0, 1, \ldots, n. \]

By Lagrange’s interpolation formula (2.1.1.4),

\[ P_n(x_i) \equiv \sum_{i=0}^{n} f_i L_i(x), \quad L_i(x) = \prod_{k=0}^{n} \frac{x - x_k}{x_i - x_k}, \]

or introducing the new variable \( t \) such that \( x = a + ht \),

\[ L_i(x) = \phi_i(t) := \prod_{k=0}^{n} \frac{t - k}{i - k}. \]

Integration gives

\[
\int_a^b P_n(x) \, dx = \sum_{i=0}^{n} f_i \int_a^b L_i(x) \, dx
\]

\[
= h \sum_{i=0}^{n} f_i \int_0^n \phi_i(t) \, dt
\]

\[
= h \sum_{i=0}^{n} f_i \alpha_i.
\]

Note that the coefficients or weights

\[ \alpha_i := \int_0^n \phi_i(t) \, dt \]

depend solely on \( n \); in particular, they do not depend on the function \( f \) to be integrated, nor on the boundaries \( a, b \) of the integral.

If \( n = 2 \) for instance, then

\[ \alpha_0 = \int_0^2 \frac{t^2 - 1}{t - 2} \, dt = \frac{1}{2} \int_0^2 (t^2 - 3t + 2) \, dt = \frac{1}{2} \left( \frac{8}{3} - \frac{12}{2} + 4 \right) = \frac{1}{3}, \]

\[ \alpha_1 = \int_0^2 \frac{t - 2}{1 - 2} \, dt = \int_0^2 (t^2 - 2t) \, dt = - \left( \frac{8}{3} - 4 \right) = \frac{4}{3}, \]

\[ \alpha_2 = \int_0^2 \frac{t - 1}{1 - 2} \, dt = \frac{1}{2} \int_0^2 (t^2 - t) \, dt = \frac{1}{2} \left( \frac{8}{3} - \frac{4}{2} \right) = \frac{1}{3}. \]
and we obtain the following approximate value: 

\[ \int_a^b P_2(x) \, dx = \frac{h}{3}(f_0 + 4f_1 + f_2) \]

for the integral \( \int_a^b f(x) \, dx \). This is Simpson’s rule.

For any natural number \( n \), the Newton-Cotes formulas

(3.1.1) \[ \int_a^b P_n(x) \, dx = h \sum_{i=0}^n f_i \alpha_i, \quad f_i = f(a + ih), \quad h := \frac{b - a}{n}, \]

provide approximate values for \( \int_a^b f(x) \, dx \). The weights \( \alpha_i, \ i = 0, 1, \ldots, n \), have been tabulated. They are rational numbers with the property

(3.1.2) \[ \sum_{i=0}^n \alpha_i = n. \]