Fast computation of the zeros of a polynomial

David S. Watkins

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Washington State University

Valencia, 2012

Collaborators

This is joint work with

- **Raf Vandebril** (KU Leuven)
- **Jared Aurentz** (WSU)

The Problem

- $p(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n = 0$
- Why on Earth?
- companion matrix

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \quad 0 \end{bmatrix}$$

- Get the zeros of p by computing the eigenvalues of A.

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Fiedler factorization (by Gaussian elimination)

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & & & \\ & 1 & & \\ & & 1 & 0 \end{bmatrix}$$

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Fiedler factorization (by Gaussian elimination)

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$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ 1 & & & \\ & 1 & & \\ & & 1 & 0 \end{bmatrix} = \quad \begin{matrix} \uparrow \\ \downarrow \end{matrix} \quad \begin{matrix} \uparrow \\ \downarrow \end{matrix} \quad \begin{matrix} \uparrow \\ \downarrow \end{matrix}$$

Fiedler factorization (by Gaussian elimination)

- Factors are

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- An $n \times n$ matrix has $n - 1$ factors.
- We will use this factored form.

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A variety of orderings

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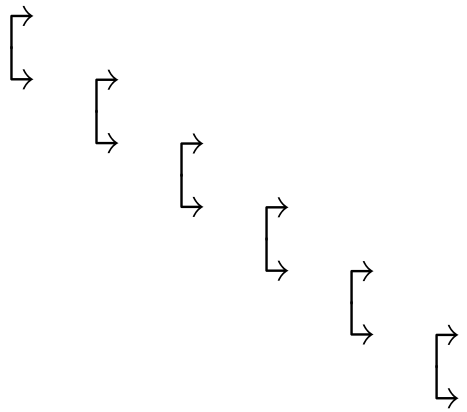
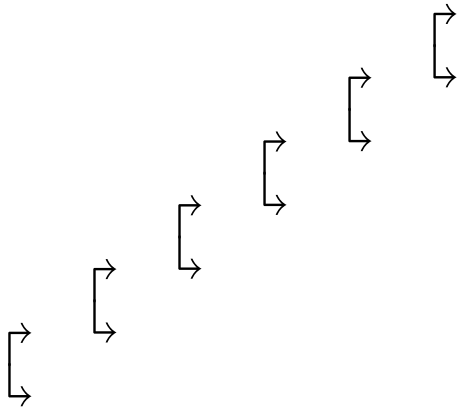
Upper Hessenberg



A variety of orderings

Lower Hessenberg



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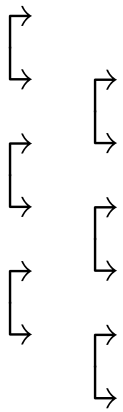
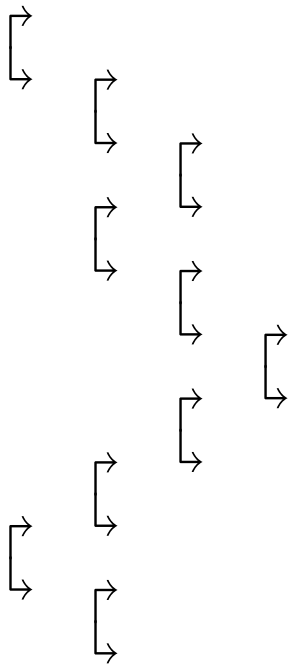
so called CMV



A variety of orderings

Random



A variety of orderings

- Our algorithms can be applied to any of these orderings.
- Factors don't have to come from a companion matrix.
- We consider upper Hessenberg case for simplicity.

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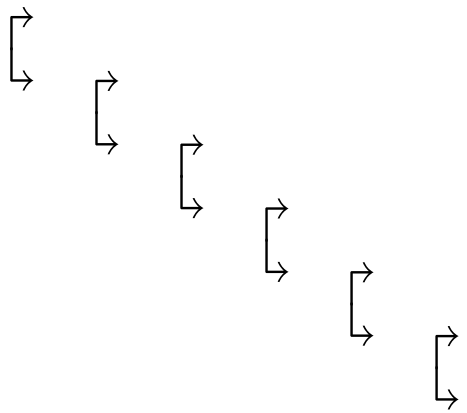
Upper Hessenberg matrix



- Matrix is stored in $O(n)$ space.
- Can we find eigenvalues in $O(n^2)$ time?

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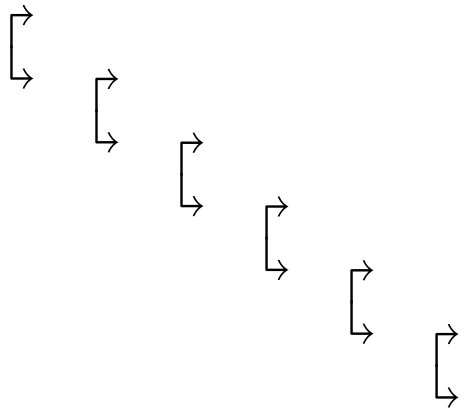

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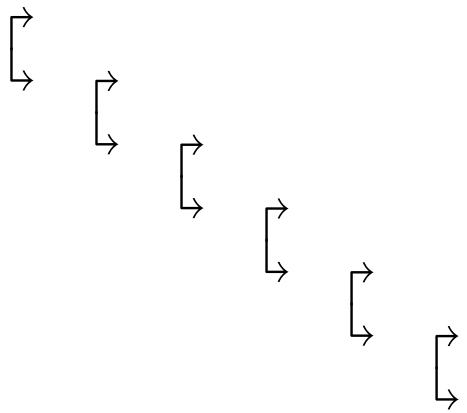

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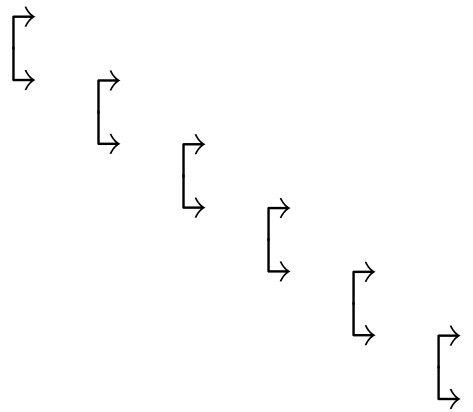
Inspiration from the unitary eigenvalue problem



- Every unitary upper Hessenberg matrix . . .
- Factors are unitary
- QR algorithm ($O(n^2)$ time)
- Gragg (1986)
- Ammar, Reichel, M. Stewart, Bunse-Gerstner, Elsner, . . .

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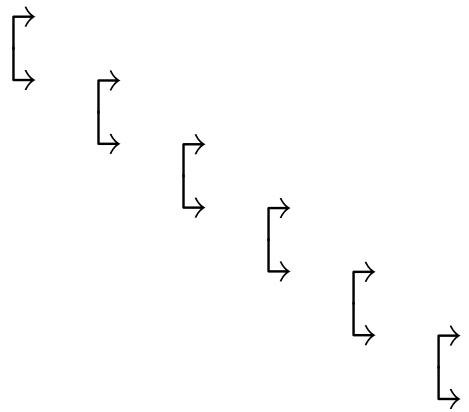

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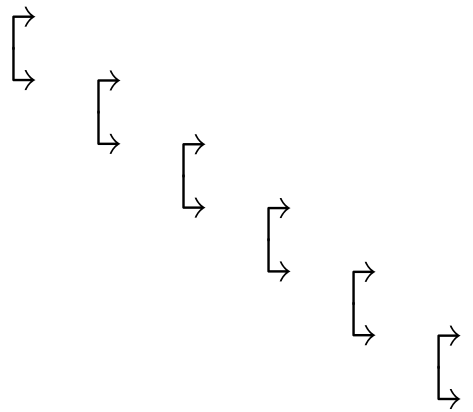

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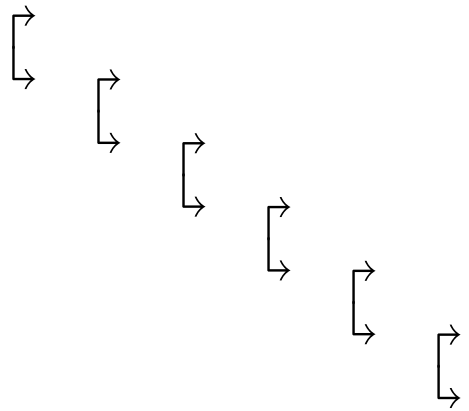

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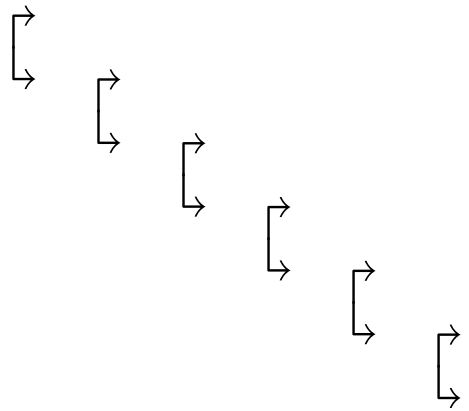

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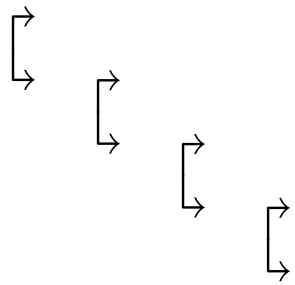
Francis algorithm on a unitary Hessenberg matrix



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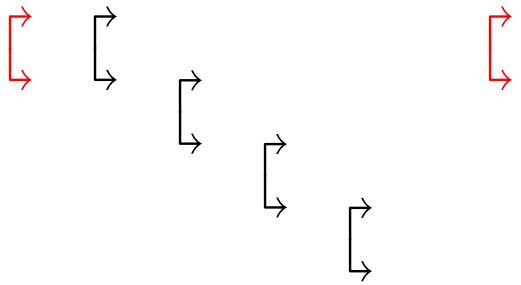

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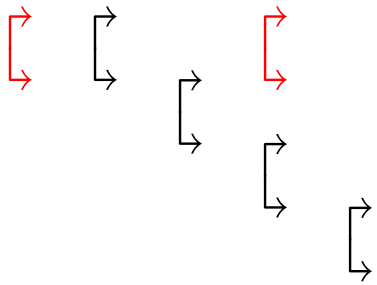

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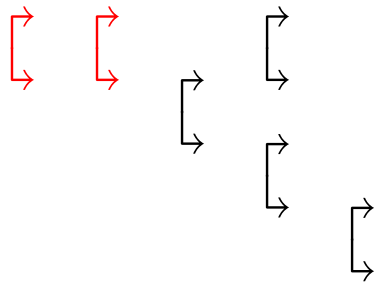

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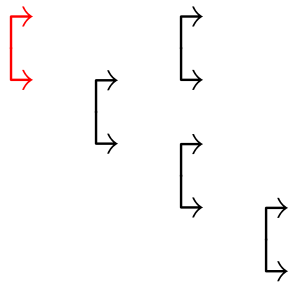

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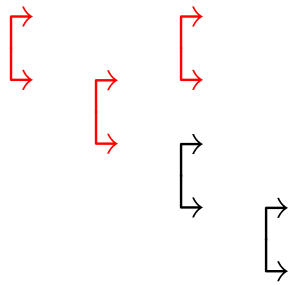

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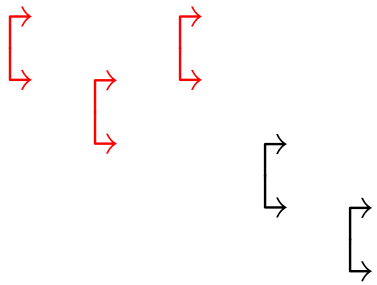

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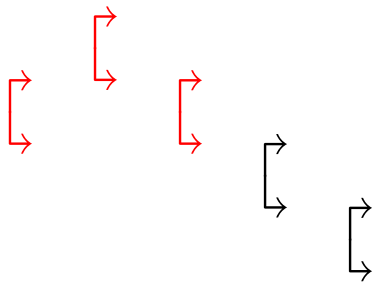

- iteration of degree one for simplicity

Allowed Operations

- fusion

- turn over

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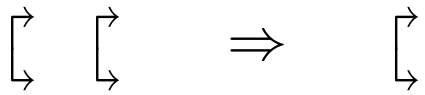

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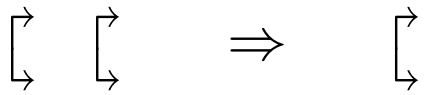

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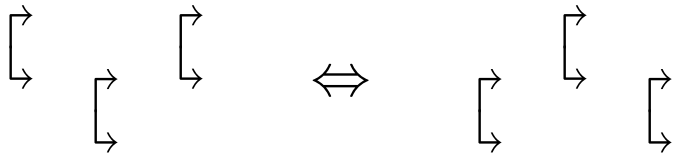

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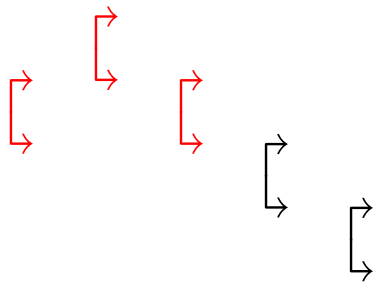

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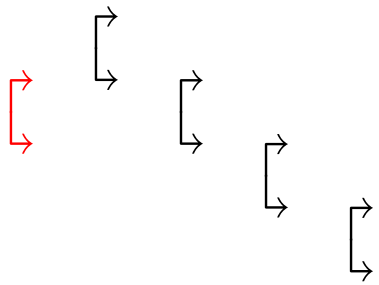

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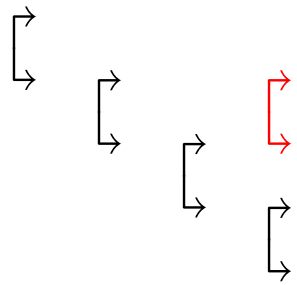

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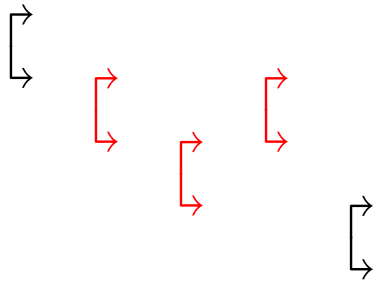

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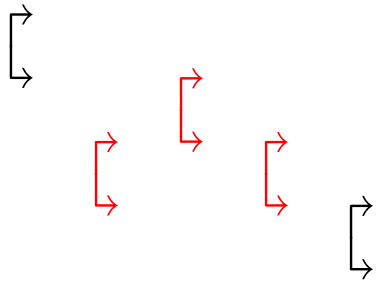

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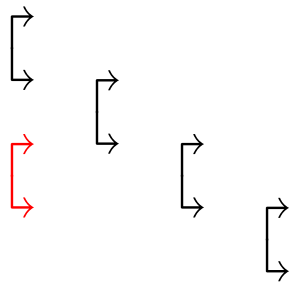

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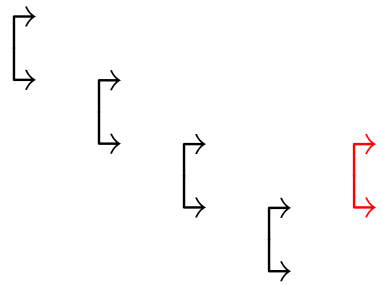

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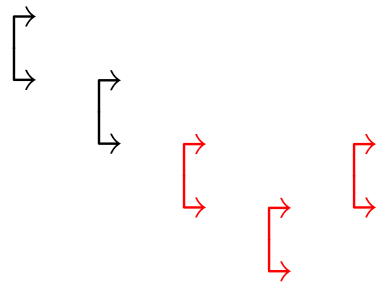

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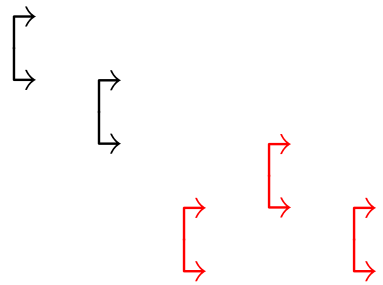

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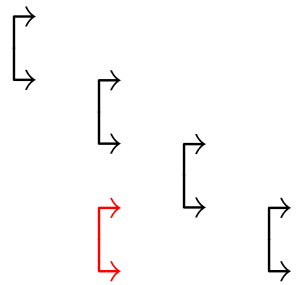

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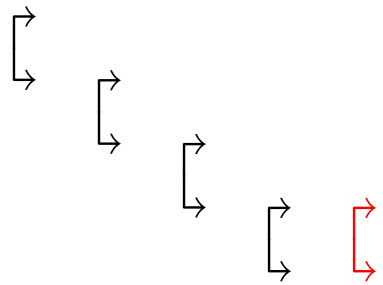

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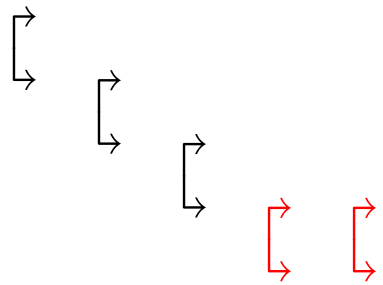

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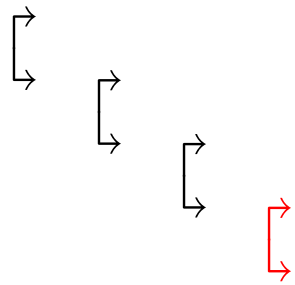

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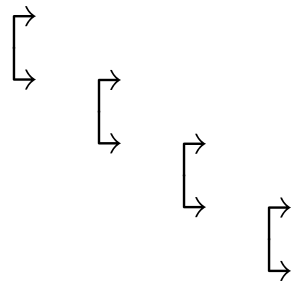

- Done!

Francis algorithm on a unitary Hessenberg matrix

- memory requirement: $O(n)$
- work per iteration: $O(n)$
- Total work: $O(n^2)$
- Can we do something like this with the companion matrix?

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Companion Matrix Factorization

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 0 \end{bmatrix} = \quad \cdots$$

- . . . but these factors are not unitary.
- Need non-unitary variant of Francis's algorithm.
- Why on Earth? Just for fun!

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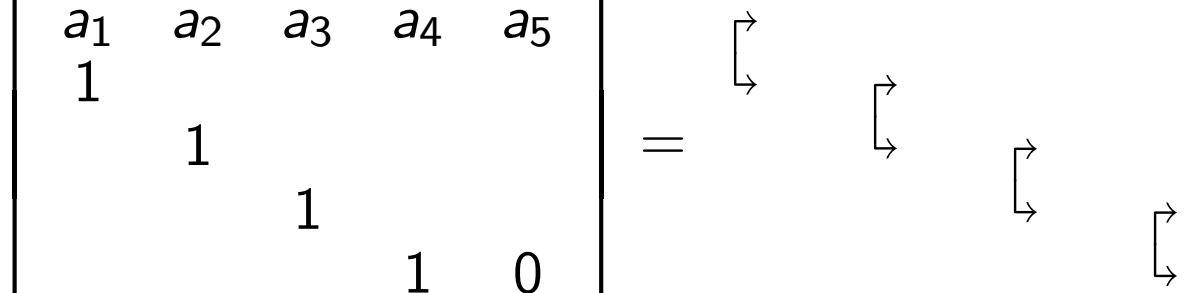

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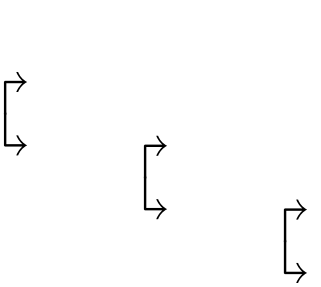

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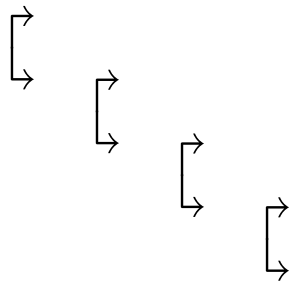
Nonunitary Francis-like algorithm (first try)



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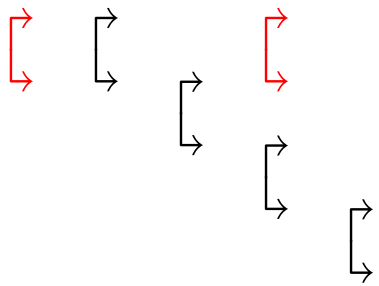

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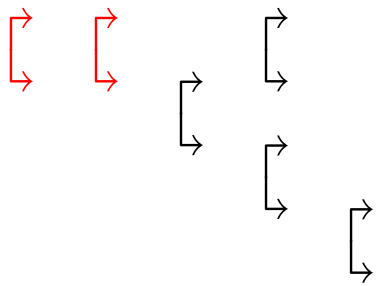

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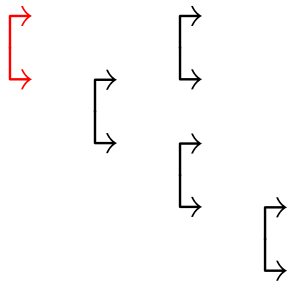

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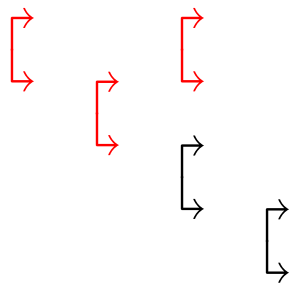

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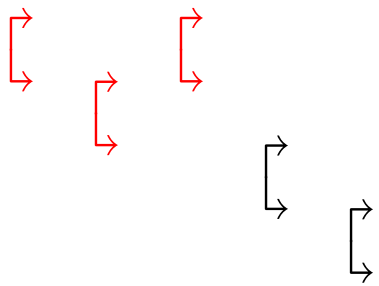

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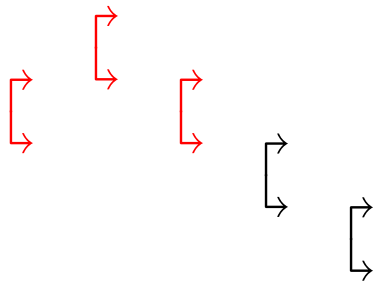
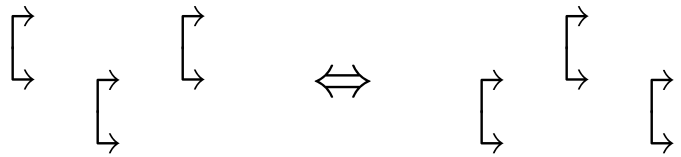


Nonunitary Francis-like algorithm (first try)

- Can we do this?



- crucial question
- Pessimistic answer: No! (not always)
- Optimistic answer: Yes! (almost always)

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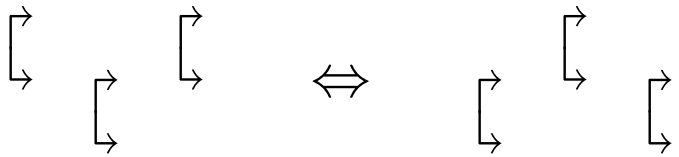

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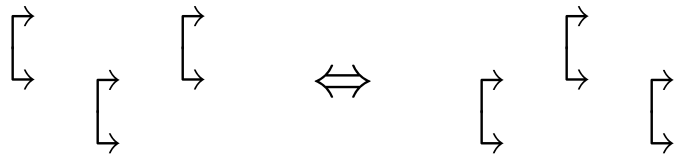

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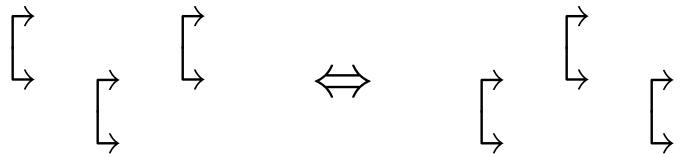

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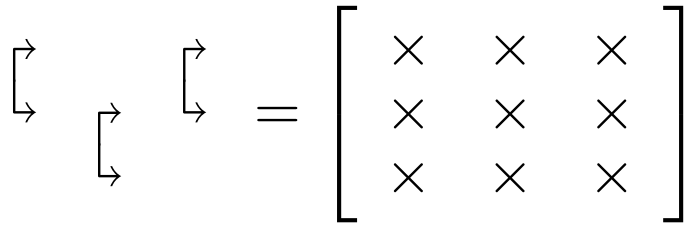
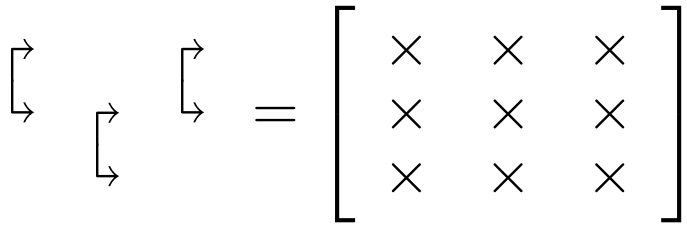
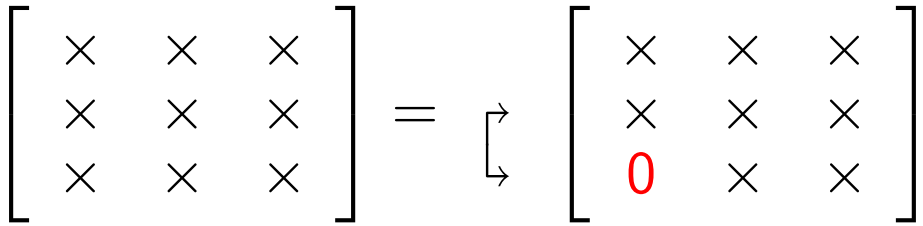
Nonunitary Turnover Operation



Need

Nonunitary Turnover Operation





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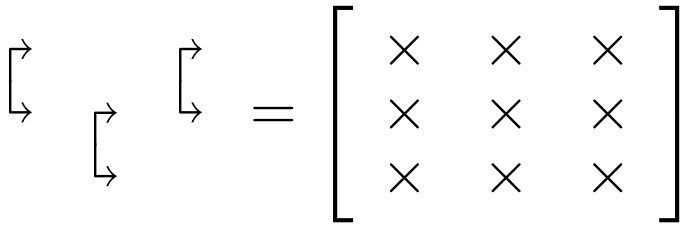

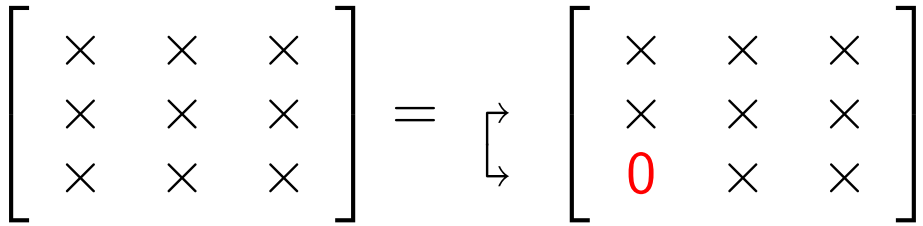

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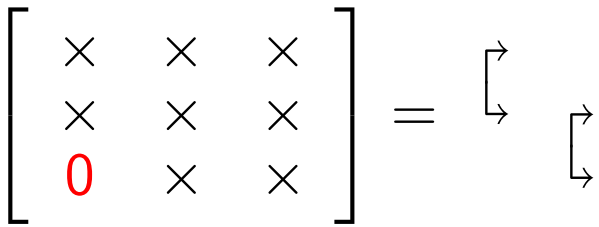

Nonunitary Turnover Operation

$$\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \end{bmatrix} = \quad \begin{matrix} \hookrightarrow \\ \hookrightarrow \end{matrix}$$

. . . is possibile iff 2×2 submatrix has rank one.

Nonunitary Turnover Operation

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Nonunitary Turnover Operation

- procedure with corrective Gauss transform
- (works almost always)

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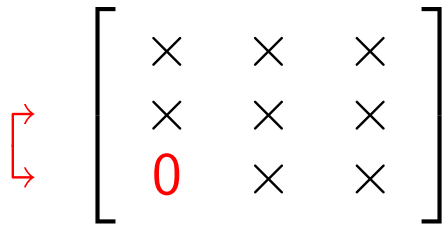

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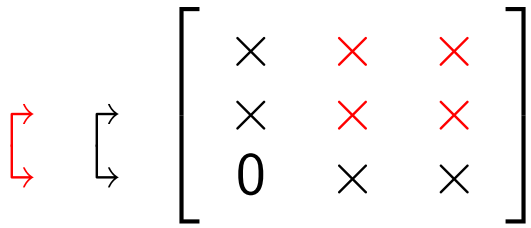

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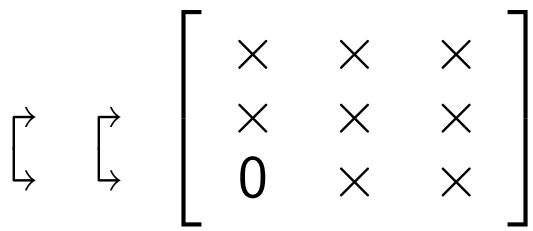

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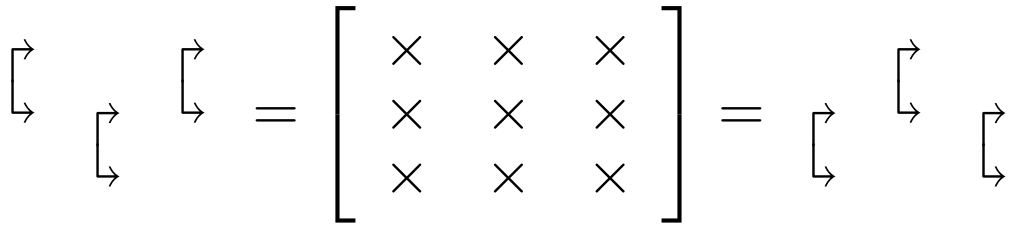
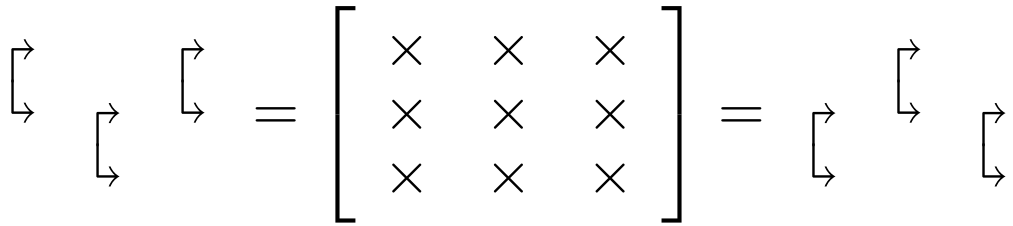
- Option: do both at once.

Nonunitary Turnover Operation



- yields a non-unitary variant of Francis's algorithm
- $O(n)$ space and $O(n^2)$ time
- it's pretty fast
- . . . but we had a better idea.

Nonunitary Turnover Operation



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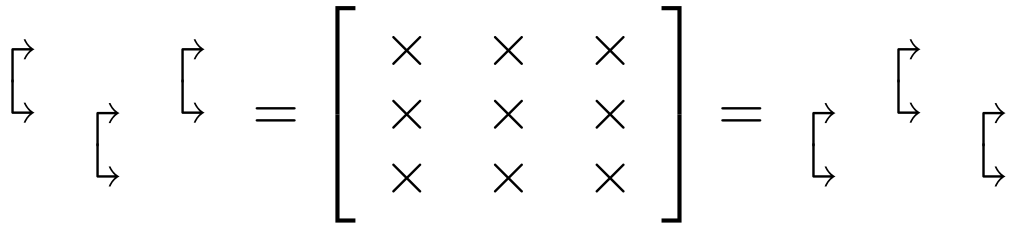

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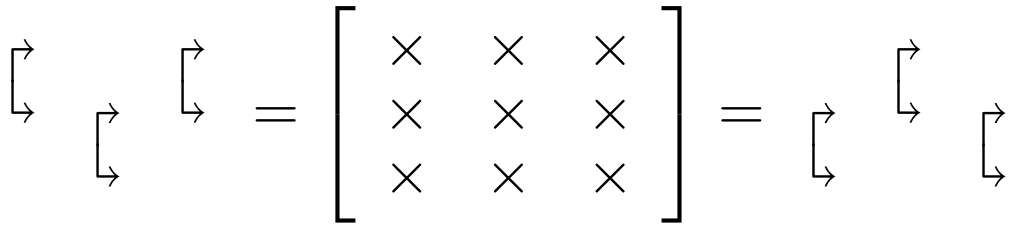

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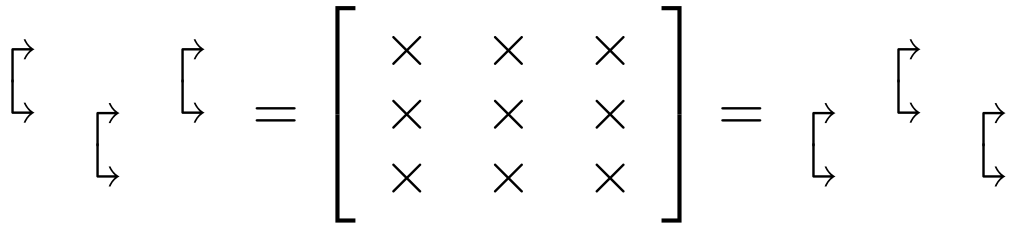

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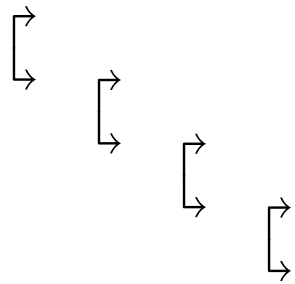
Nonunitary Francis-like algorithm (second try)



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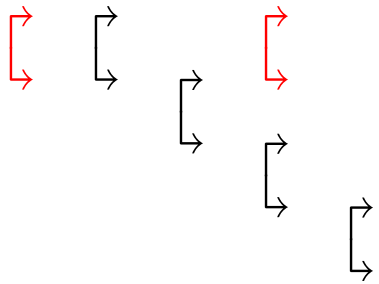

- First transformation doesn't have to be unitary.
- Use a Gauss transformation instead.
- Gauss transform = unit lower triangular
- Thanks, Jared

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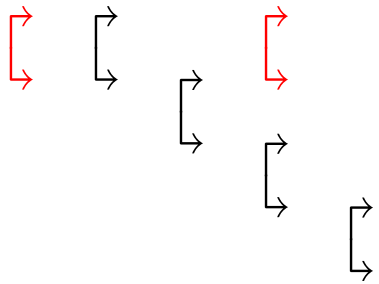

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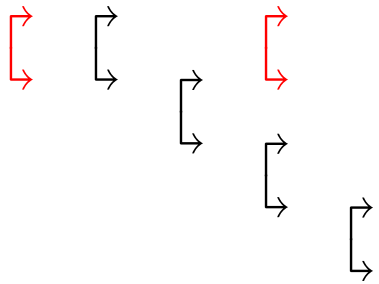

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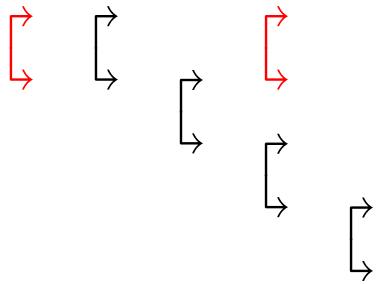

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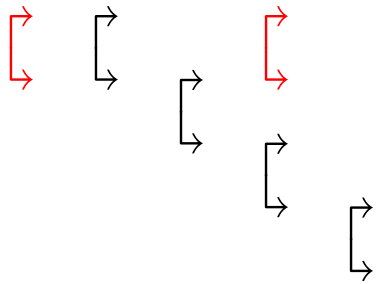

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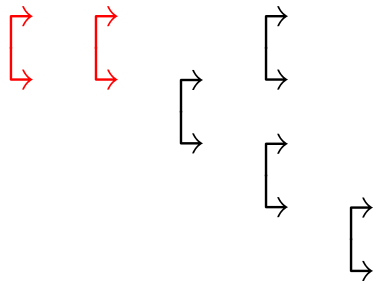

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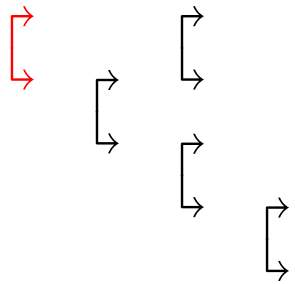

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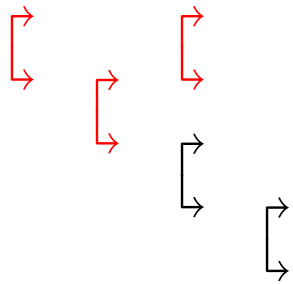

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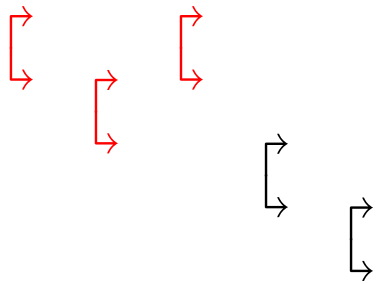

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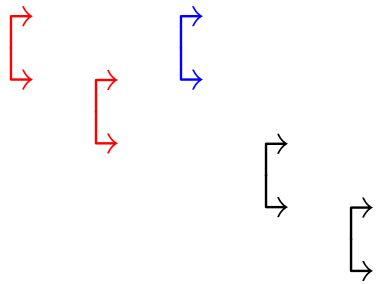
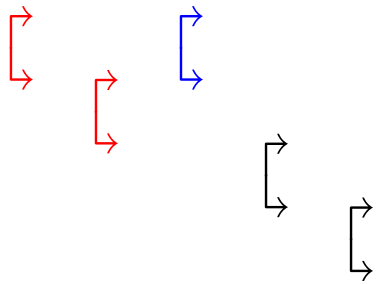


- Blue transform is a Gauss transform.

Nonunitary Francis-like algorithm (second try)



- Blue transform is a Gauss transform.
- Can we exploit this?

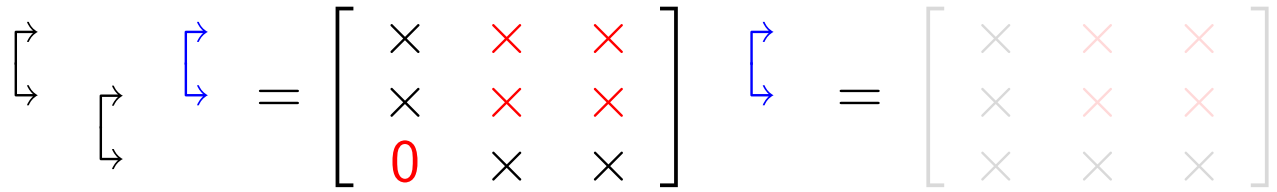
Nonunitary Francis-like algorithm (second try)

- Yes!

- submatrix still has rank one!

Nonunitary Francis-like algorithm (second try)

- Yes!



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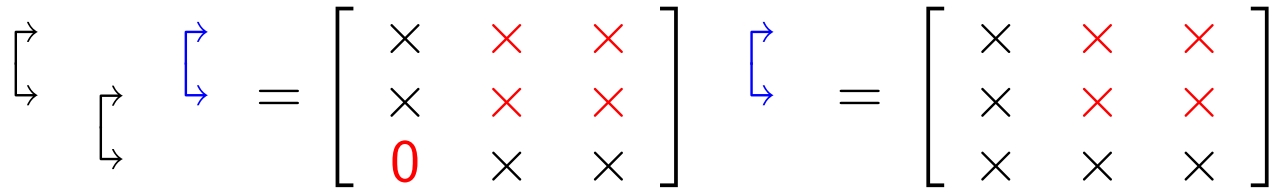


- submatrix still has rank one!

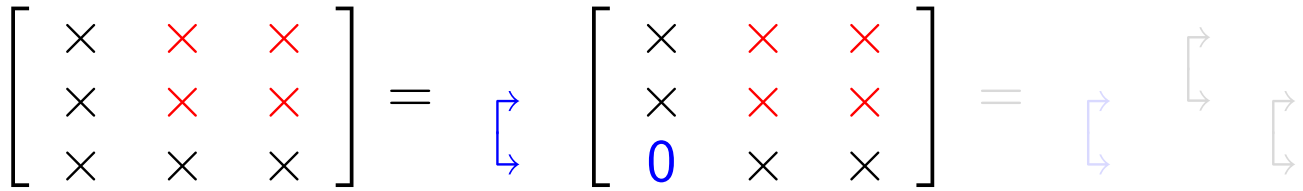
Nonunitary Francis-like algorithm (second try)

- So . . .

- (Blue) Gauss transform does not disturb 2×2 submatrix.

Nonunitary Francis-like algorithm (second try)

- So ...



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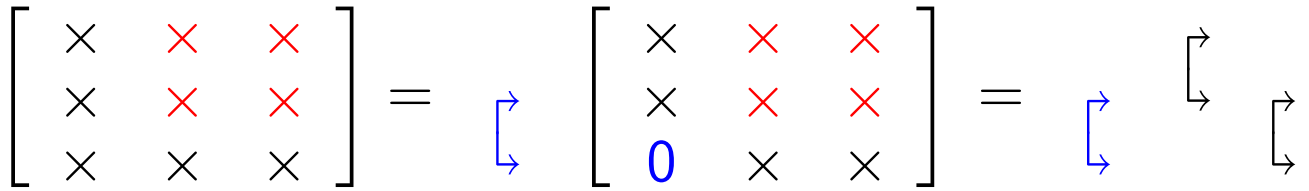
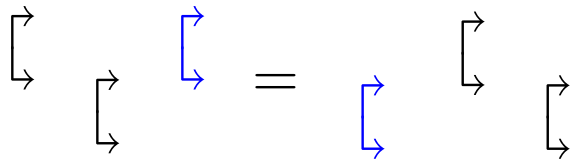


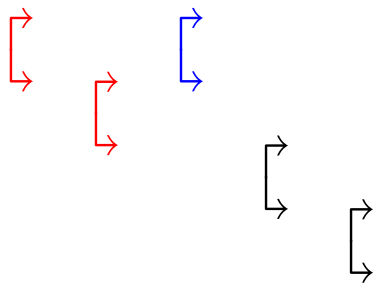
- (Blue) Gauss transform does not disturb 2×2 submatrix.

Nonunitary Francis-like algorithm (second try)

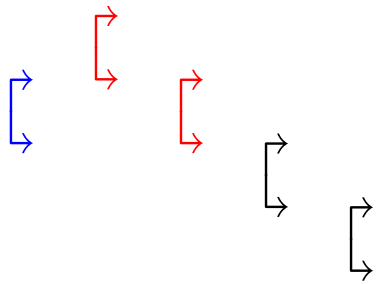
- Complete turnover looks like this:



Nonunitary Francis-like algorithm (second try)



Nonunitary Francis-like algorithm (second try)



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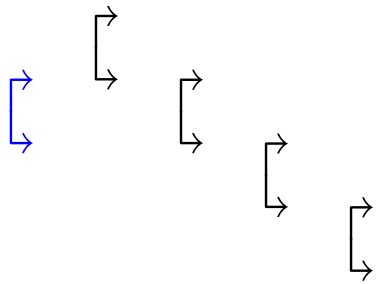

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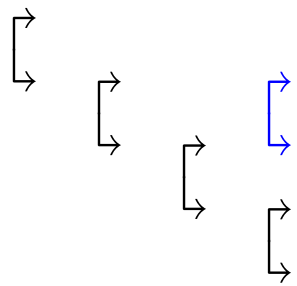

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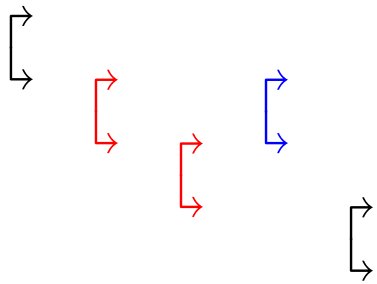

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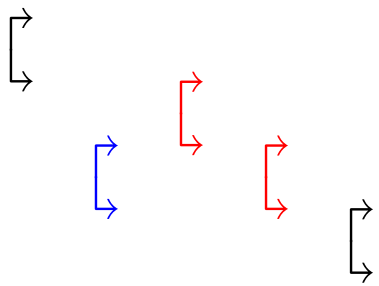

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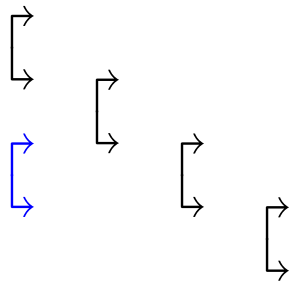

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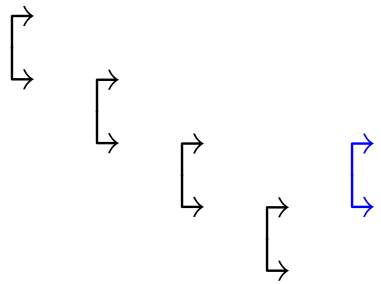

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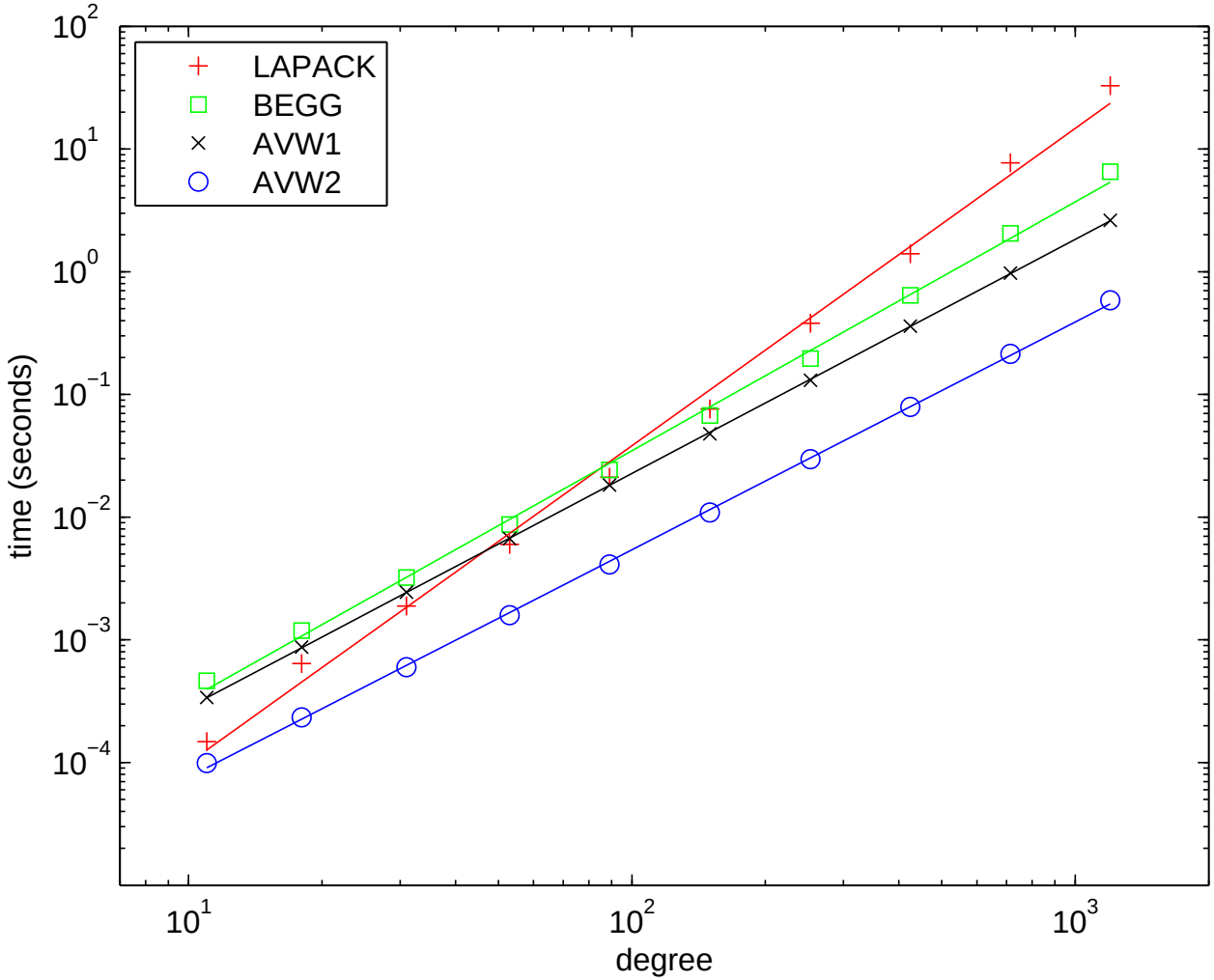
Contestants

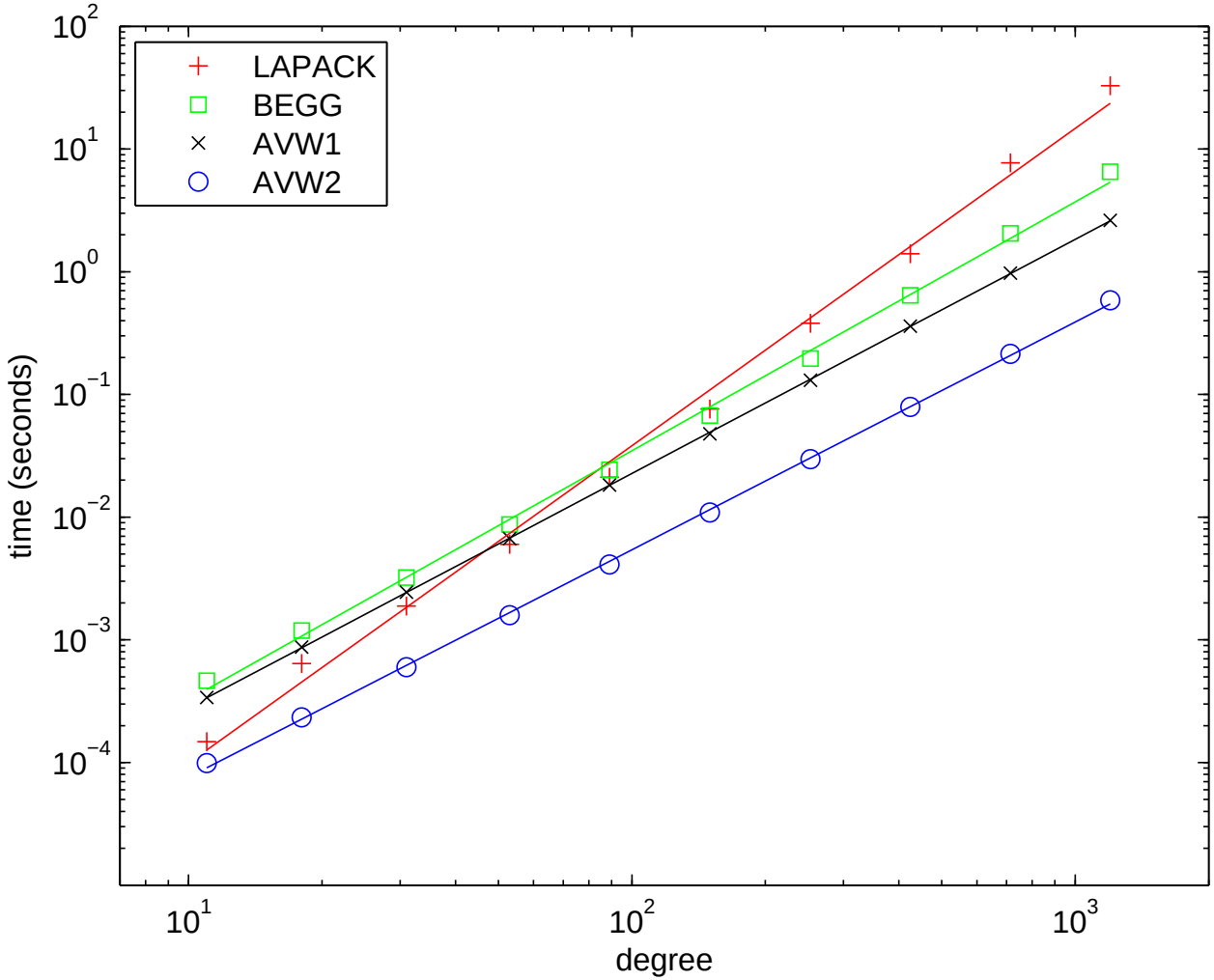
- LAPACK code ZHSEQR ($O(n^3)$)
- BBEGG (fast, unitary, one of the better codes)
- AVW1 (our first attempt)
- AVW2 (our second attempt)

Speed Comparison



No crossover point!

Speed Comparison



No crossover point!

Speed Comparison

- At $n = 1200$:

method	time
LAPACK	32.7
BBEGG	6.5
AVW1	2.6
AVW2	0.6

Final Words

- Our code is lightning fast, but . . .
- how robust is it?
- Answer: Not as robust as we'd like.
- We can't recommend it for general use, but . . .
- it must be good for something.
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Thank you for your attention.