Solving the Matrix Eigenvalue Problem: Chasing bulges, cores, or poles

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An Old Topic

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- $A \in \mathbb{C}^{n \times n}$

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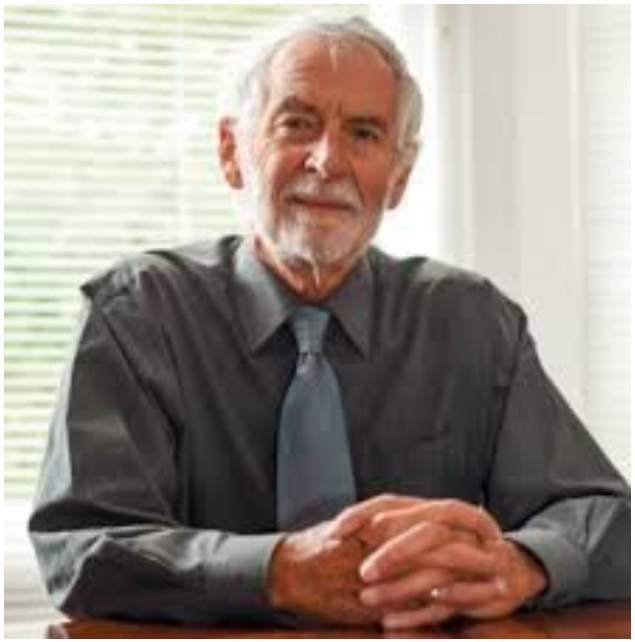
- Haven't these problems been solved?

The Winning Algorithm

John Francis



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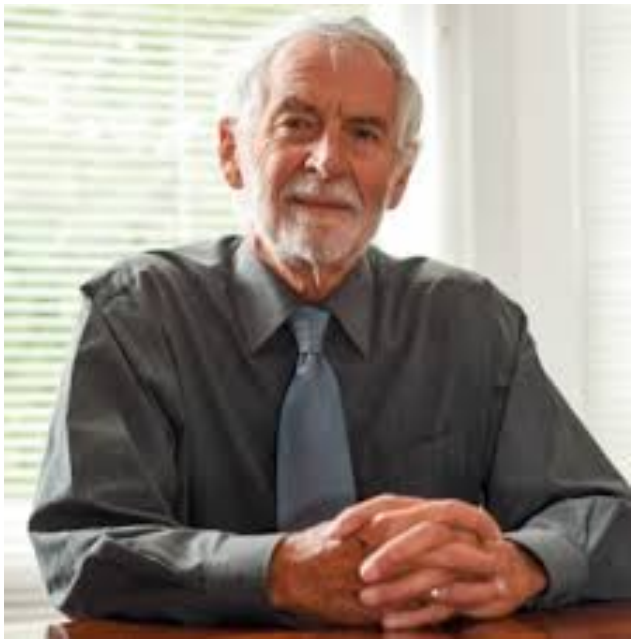

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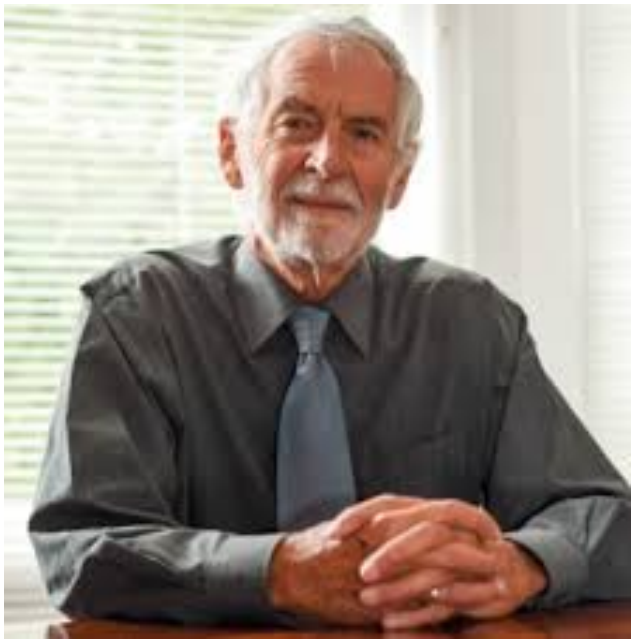

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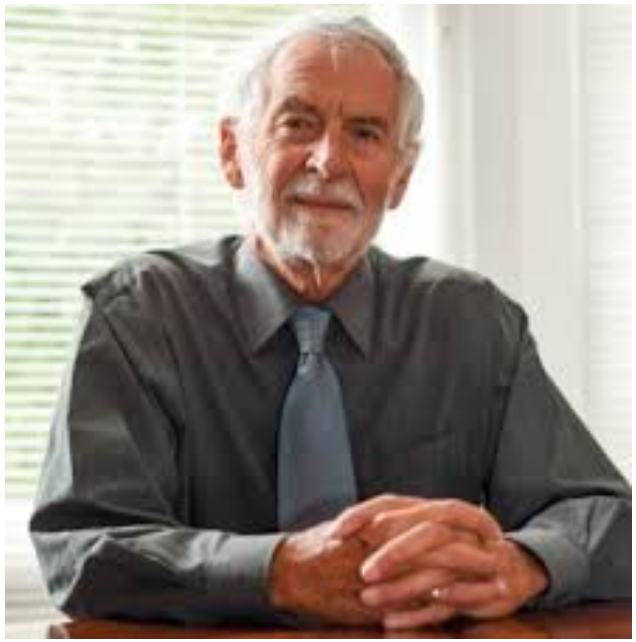

- invented the winning algorithm in 1959.
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- I call it Francis's algorithm.

The Winning Algorithm

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This is an old topic, but ...

Proliferation of Possibilities

Ways to implement Francis's algorithm:

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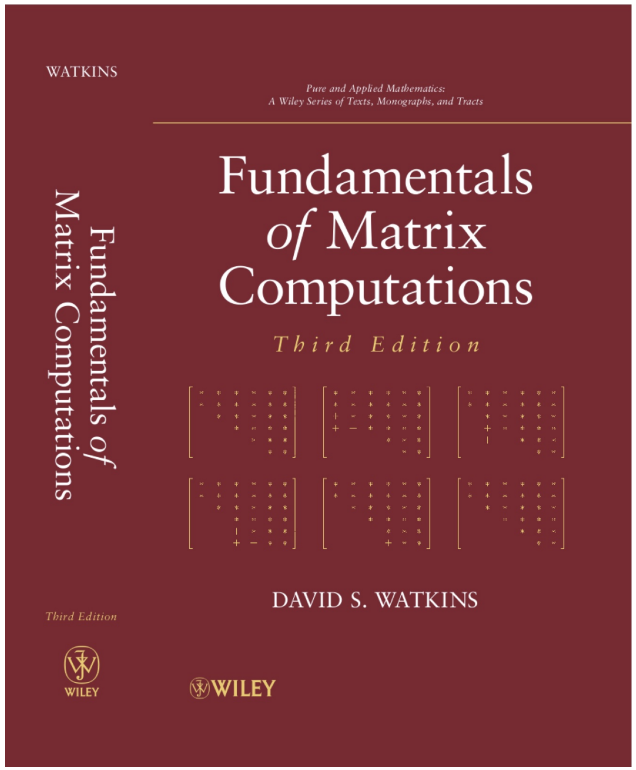

Proliferation of Possibilities

- core chasing (2011–)

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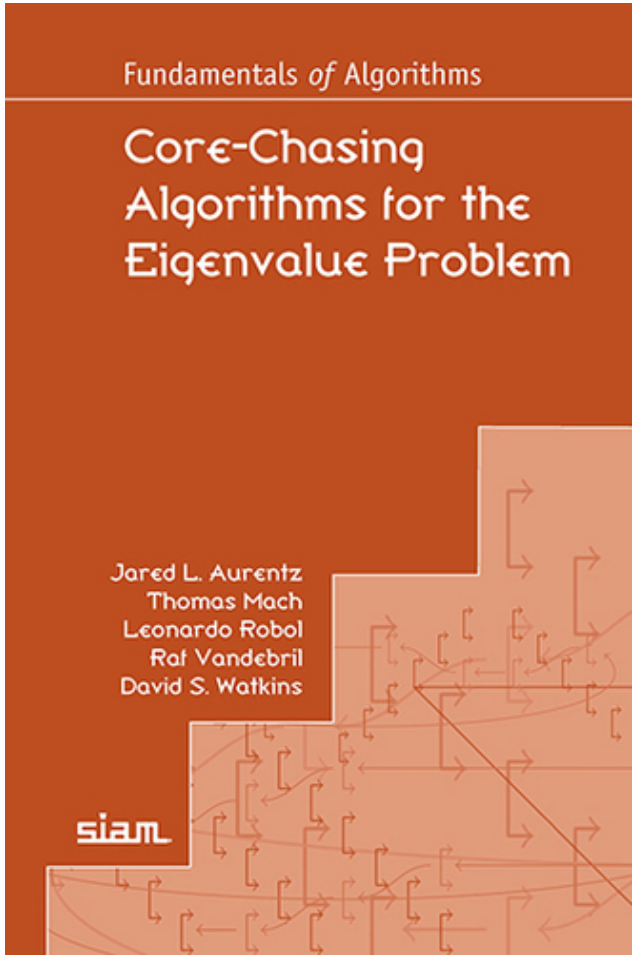

2018

Proliferation of Possibilities

- pole swapping (2016–)

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- D. Camps, K. Meerbergen, and R. Vandebril, *A rational QZ method*, SIAM J. Matrix Anal. Appl., 40 (2019) pp. 943–972.
- D. Camps, K. Meerbergen, and R. Vandebril, *A multishift, multipole rational QZ method with aggressive early deflation*, arXiv:1902.10954.
- D. Camps, T. Mach, R. Vandebril, and D. S. Watkins, *On pole-swapping algorithms for the eigenvalue problem*, arXiv:1906.08672.

Proliferation of Possibilities

In summary: There are at least three ways to implement Francis and related algorithms.

- bulge chasing
- core chasing
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In summary: There are at least three ways to implement Francis and related algorithms.

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. . . and some of them are brand new!

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- single-shift algorithm (not double-shift)

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- upper Hessenberg

$$A = \begin{bmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & \times \\ & & \times & \times & \times & \times \\ & & & \times & \times & \times \\ & & & & \times & \times \end{bmatrix}$$

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- iterate!

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Description of one iteration.

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- Compute $x = (A - \rho I)e_1 = \begin{bmatrix} a_{11} - \rho \\ a_{21} \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \times \\ \times \\ 0 \\ \vdots \end{bmatrix}$.

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- Similarity transformation $A \to Q^* A Q$

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- Similarity transformation $A \to Q^* A Q$ creates a bulge in the Hessenberg form.

Francis Bulge-Chasing Algorithm

Q^*AQ

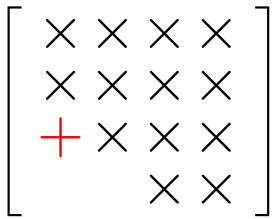
Francis Bulge-Chasing Algorithm

$$Q^*AQ = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ & \times & \times & \times \\ & & \times & \times \end{bmatrix}$$

Francis Bulge-Chasing Algorithm

$$Q^*AQ = \hookrightarrow \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ & \times & \times & \times \\ & & \times & \times \end{bmatrix} \hookrightarrow = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ + & \times & \times & \times \\ & & \times & \times \end{bmatrix}$$

Francis Bulge-Chasing Algorithm



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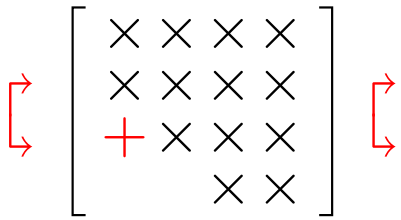

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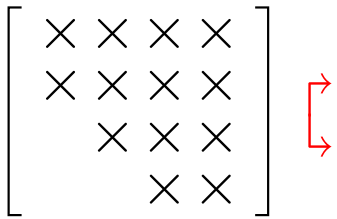

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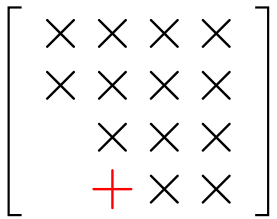

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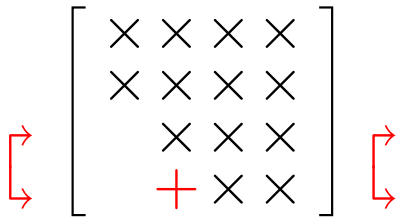

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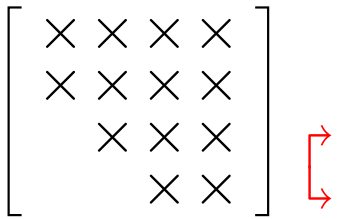

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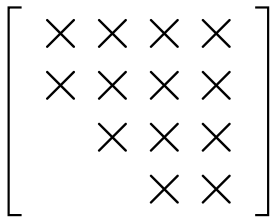

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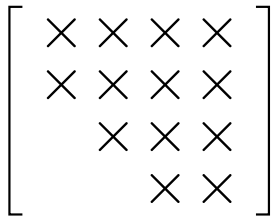

- $a_{n,n-1} \to 0$ (quadratically)

Francis's Algorithm as a Core-Chasing Algorithm

Reformulation as a core-chasing algorithm

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Reformulation as a core-chasing algorithm

Write A in QR decomposed form: $A = QR$

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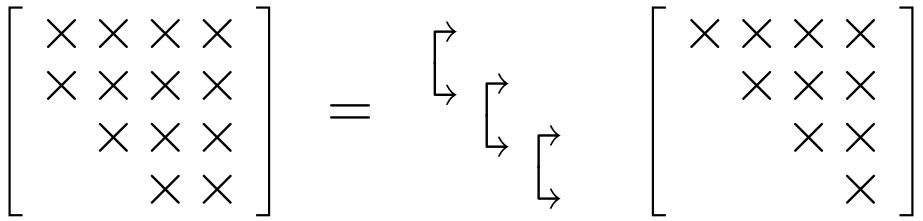

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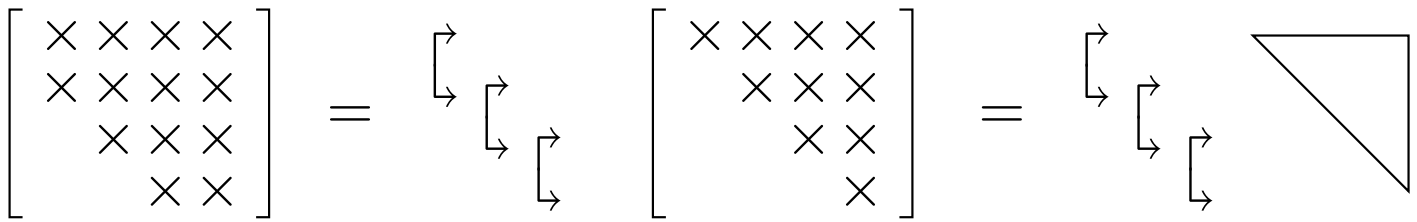
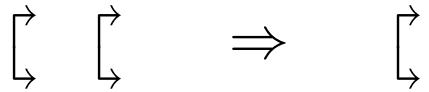


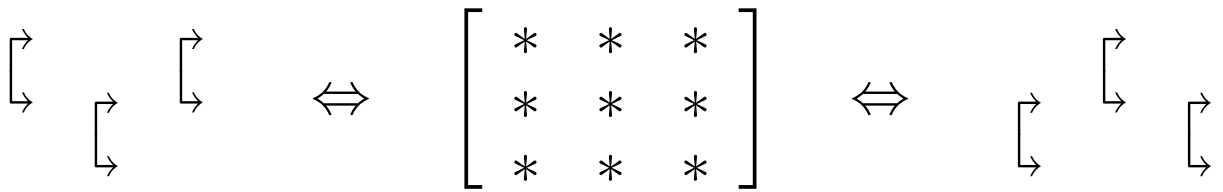
Operating on Core Transformations

Fusion



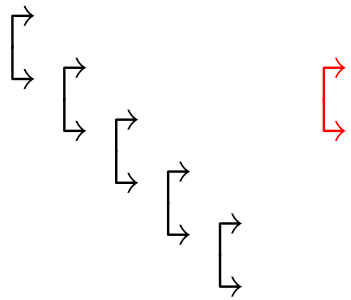
Operating on Core Transformations

Turnover



Operating on Core Transformations

Turnover as a shift-through operation



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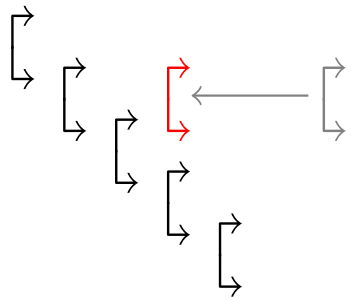

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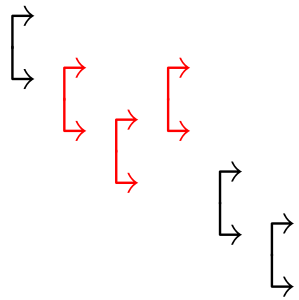

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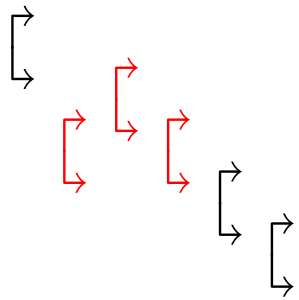

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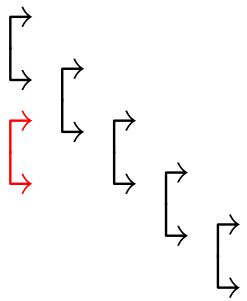

Operating on Core Transformations

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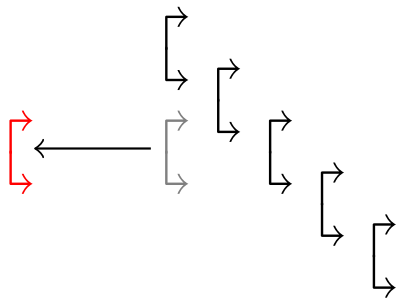

Operating on Core Transformations

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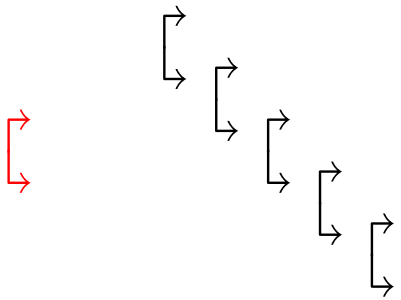
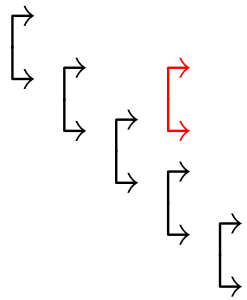


Operating on Core Transformations

Turnover as a shift-through operation : Abbreviated notation



Operating on Core Transformations

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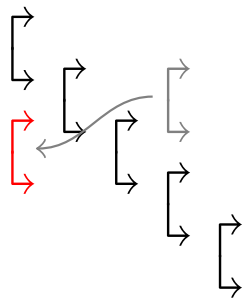
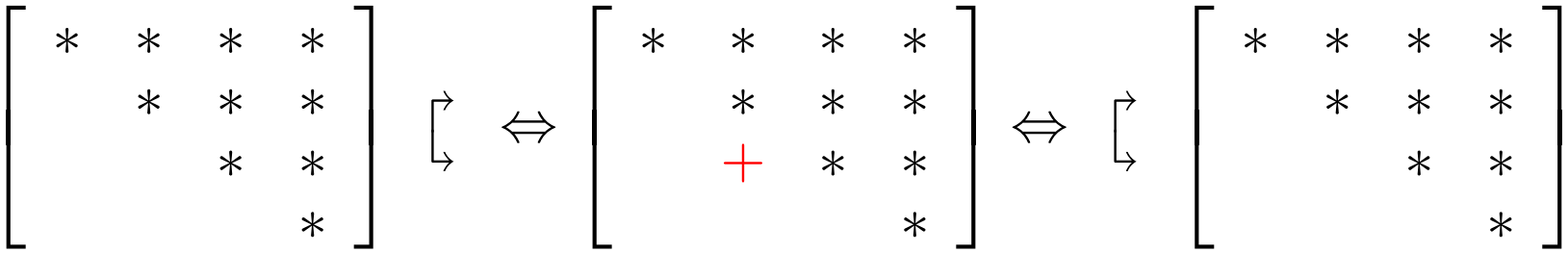


Operating on Core Transformations

Passing a core transformation through a triangular matrix

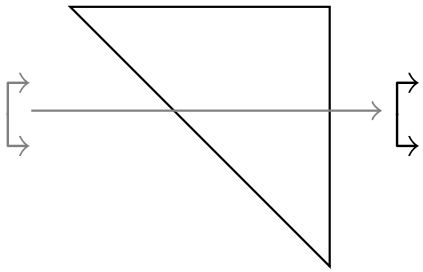


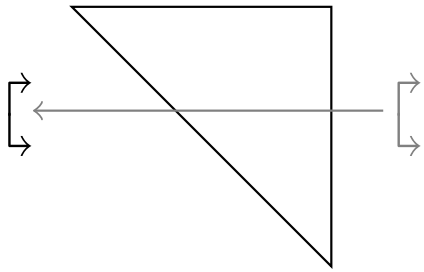
Cost is $O(n)$ flops.

Operating on Core Transformations

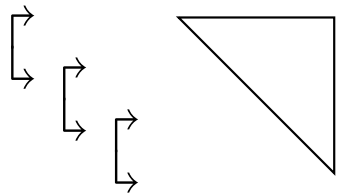
Passing a core transformation through a triangular matrix

Abbreviated Notation:





Francis's Algorithm as a Core-Chasing Algorithm

$A = QR =$ 

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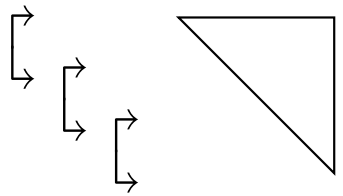
$A = QR =$ 

We will demonstrate one iteration.

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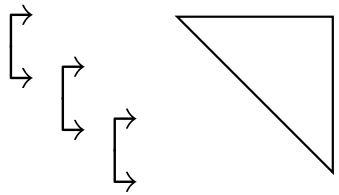

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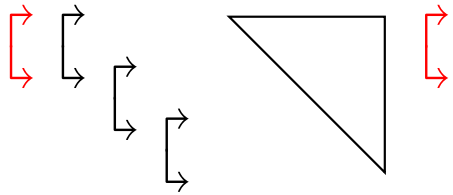

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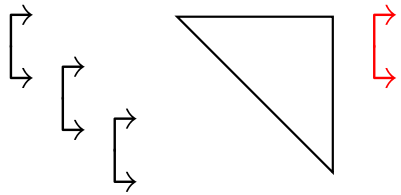
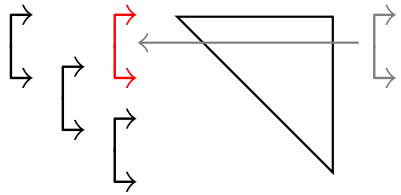


Francis's Algorithm as a Core-Chasing Algorithm



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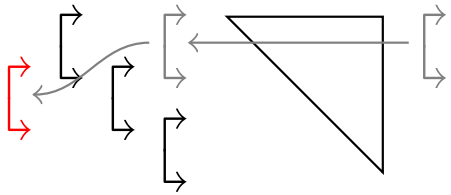

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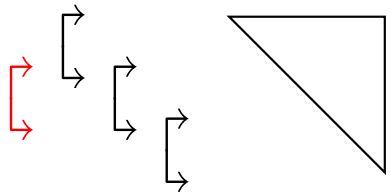

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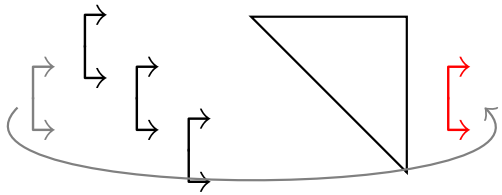

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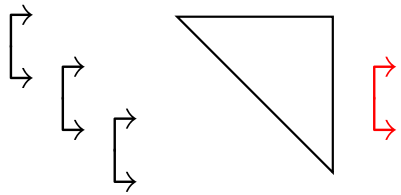

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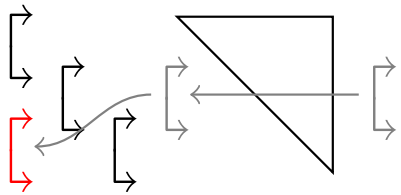

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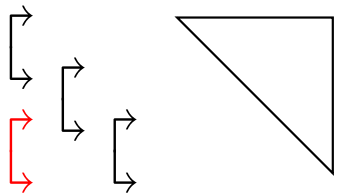

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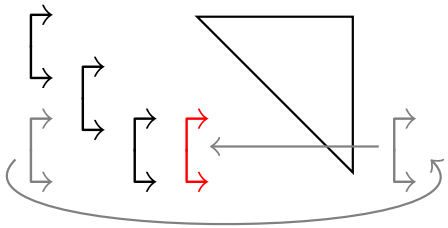

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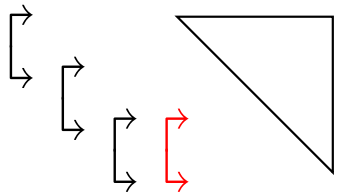

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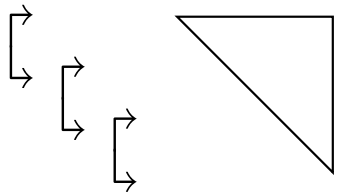

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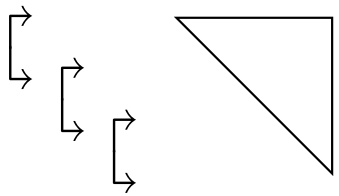

Done!

Advantages of Core Chasing

Are there any advantages?

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  - unitary-plus-low-rank

Unitary Case

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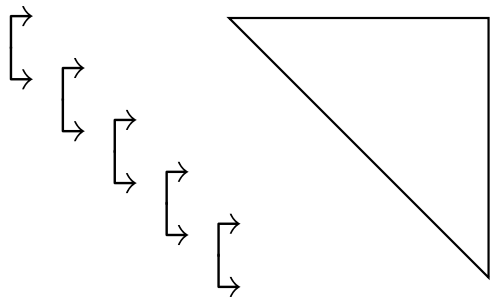
$A = QR =$ 

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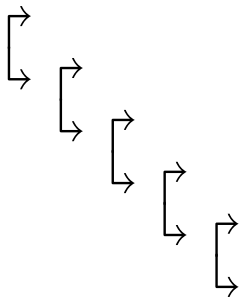
$A = QR =$ 

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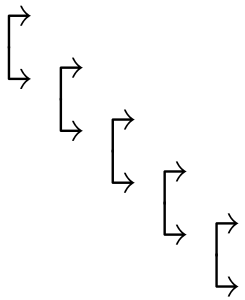
$A = QR =$ 

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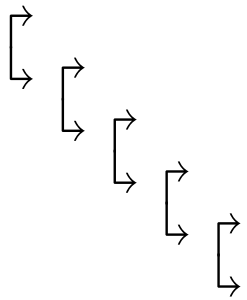
$A = QR =$ 

- The expensive part is gone!

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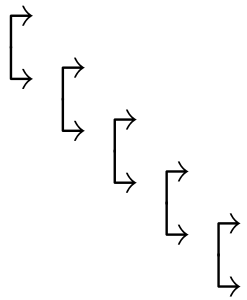
$A = QR =$ 

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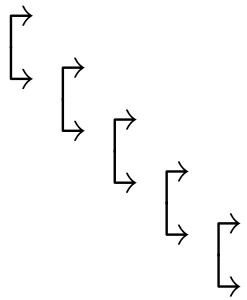
Unitary Case

$A = QR =$ 

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- Gragg (1986)

Companion Case

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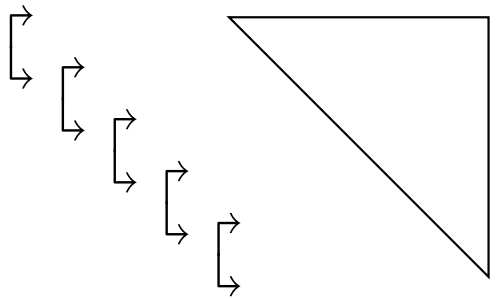
- MATLAB's `roots` command
- Companion matrix is unitary-plus-rank-one.

Unitary-plus-low-rank Case

$A = QR =$ 

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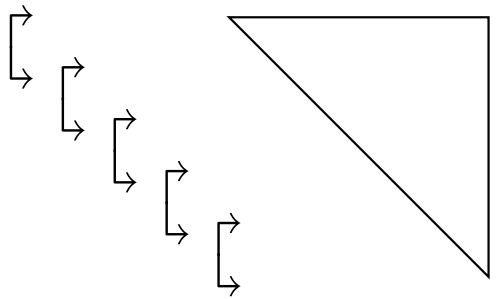
Unitary-plus-low-rank Case

$A = QR =$ 

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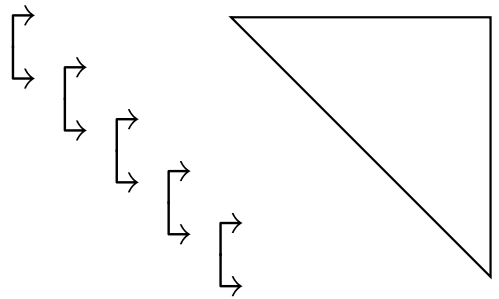
$A = QR =$ 

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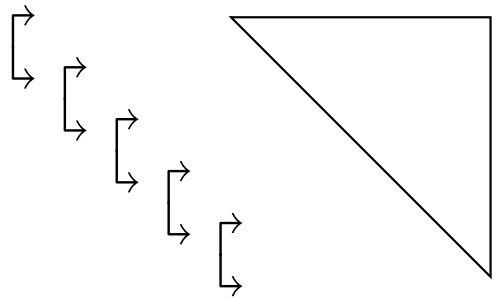
$A = QR =$ 

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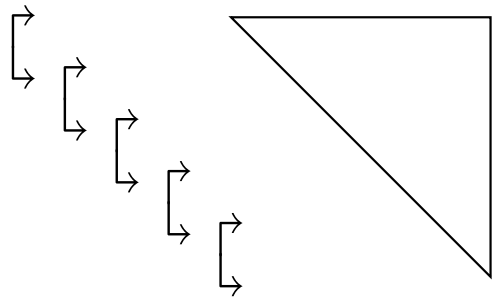
$A = QR =$ 

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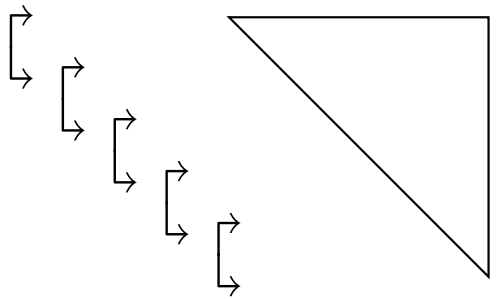
$A = QR =$ 

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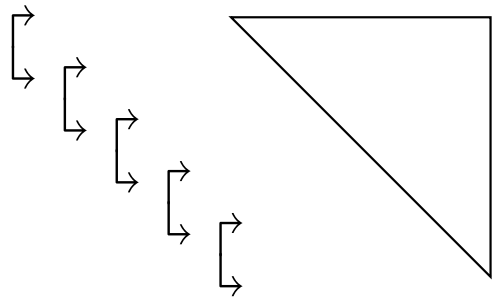
$A = QR =$ 

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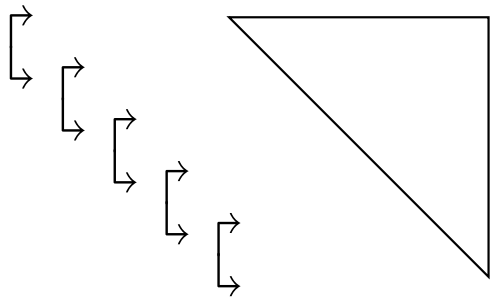
$A = QR =$ 

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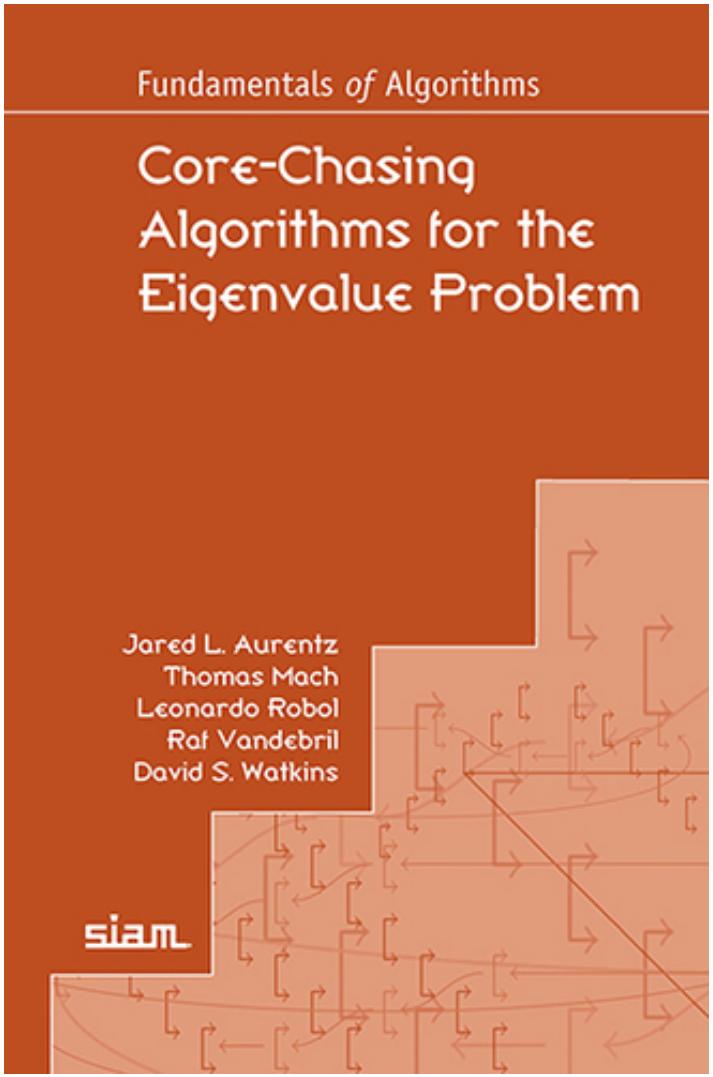
Core-Chasing Algorithms . . . (2018)



and now for Pole Swapping

So far we've looked at

and now for Pole Swapping

So far we've looked at

- bulge chasing,

and now for Pole Swapping

So far we've looked at

- bulge chasing,
- core chasing.

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Now it's time for

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This is best introduced via the generalized eigenvalue problem.

Generalized Eigenvalue Problem

$$Ax = \lambda Bx$$

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Moler/Stewart:

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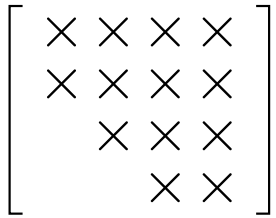
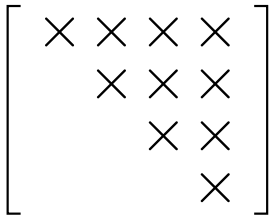
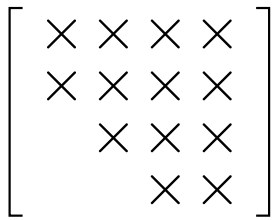
Hessenberg-triangular form

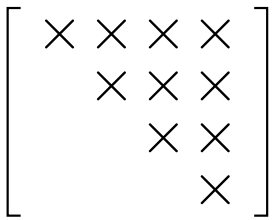
Moler/Stewart Bulge Chase





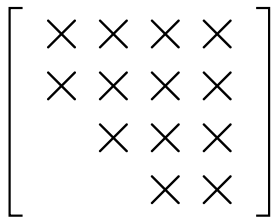
Moler/Stewart Bulge Chase

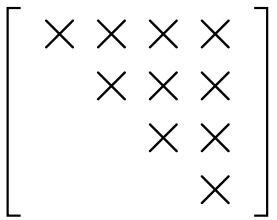




- Create a bulge and chase it.

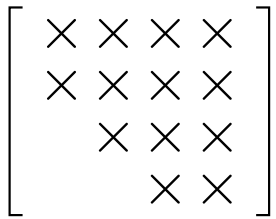
Moler/Stewart Bulge Chase

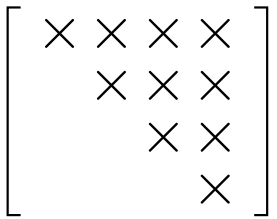




- Create a bulge and chase it.
- single-shift case

Moler/Stewart Bulge Chase

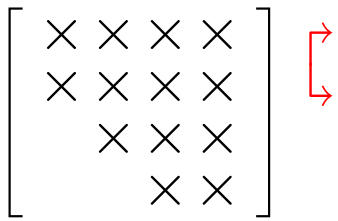


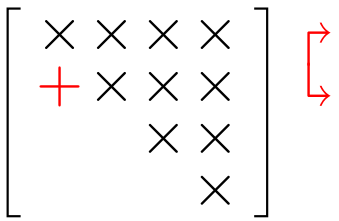


Moler/Stewart Bulge Chase

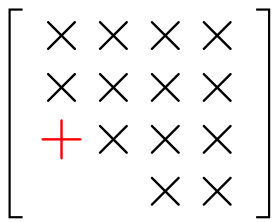
Moler/Stewart Bulge Chase

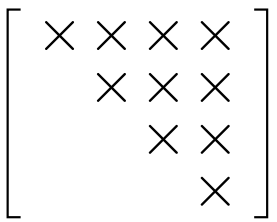
Moler/Stewart Bulge Chase





Moler/Stewart Bulge Chase

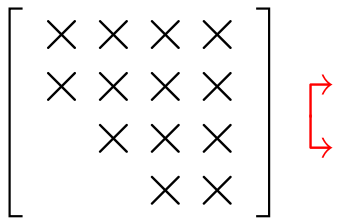


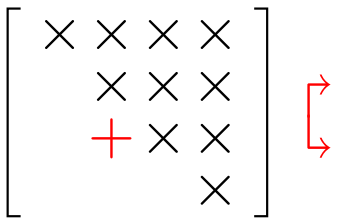


Moler/Stewart Bulge Chase

Moler/Stewart Bulge Chase

Moler/Stewart Bulge Chase





Moler/Stewart Bulge Chase

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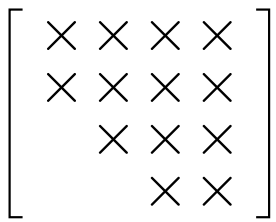
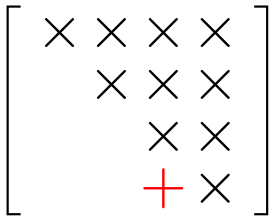
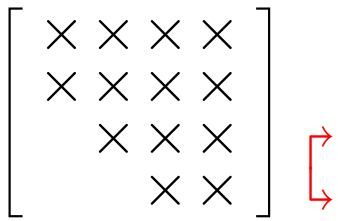
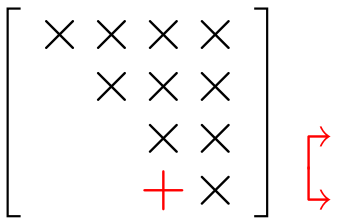
Moler/Stewart Bulge Chase



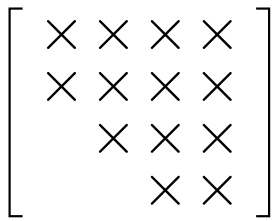


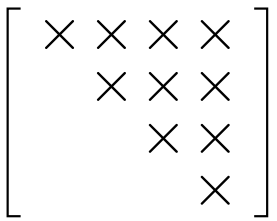
Moler/Stewart Bulge Chase



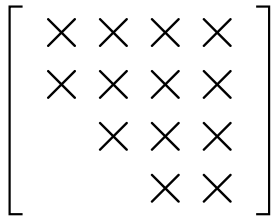


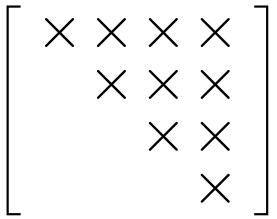
Moler/Stewart Bulge Chase





Moler/Stewart Bulge Chase





- Done!

Moler/Stewart Bulge Chase

- Done!
- Now repeat, again and again.

Moler/Stewart Bulge Chase

- Done!
- Now repeat, again and again.
- $a_{n,n-1} \to 0$ (quadratically)

Moler/Stewart Bulge Chase

- Done!
- Now repeat, again and again.
- $a_{n,n-1} \to 0$ (quadratically)

- This can also be done by core chasing,

Moler/Stewart Bulge Chase

- Done!
- Now repeat, again and again.
- $a_{n,n-1} \to 0$ (quadratically)

- This can also be done by core chasing,
- but let's go straight to pole swapping.

Hessenberg Pairs

Need a condensed form.

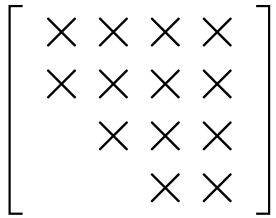
Hessenberg Pairs

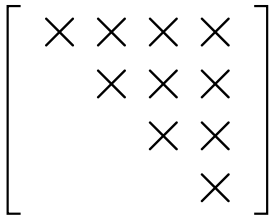
Need a condensed form.

Hessenberg-triangular form can be used, but . . .

Hessenberg Pairs

Need a condensed form.





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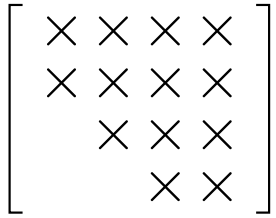
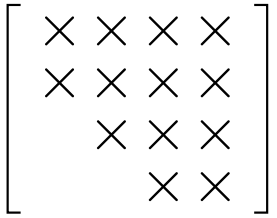
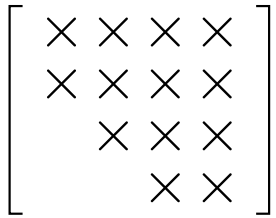


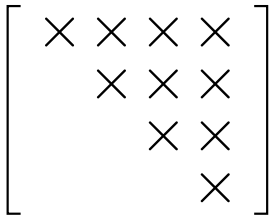


Hessenberg-Hessenberg form is good enough!

Hessenberg Pairs

Need a condensed form.





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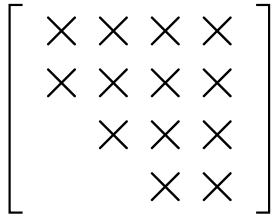

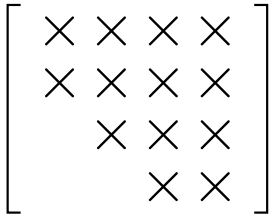

Hessenberg-Hessenberg form is good enough!

We call this a *Hessenberg pair* or *Hessenberg pencil*.

Moler/Stewart Bulge Chase Revisited

Hessenberg pair

Moler/Stewart Bulge Chase Revisited

not Hessenberg

Moler/Stewart Bulge Chase Revisited

Hessenberg pair

Moler/Stewart Bulge Chase Revisited

not Hessenberg

Moler/Stewart Bulge Chase Revisited

Hessenberg pair

Pole Swapping

Poles of a Hessenberg pair

Pole Swapping

Poles of a Hessenberg pair

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \alpha_1 & \times & \times & \times \\ & \alpha_2 & \times & \times \\ & & \alpha_3 & \times \end{bmatrix} \qquad B = \begin{bmatrix} \times & \times & \times & \times \\ \beta_1 & \times & \times & \times \\ & \beta_2 & \times & \times \\ & & \beta_3 & \times \end{bmatrix}$$

Pole Swapping

Poles of a Hessenberg pair

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Poles: α_1/β_1 α_2/β_2 α_3/β_3

Pole Swapping

Poles of a Hessenberg pair

$$A = \begin{bmatrix} \times & \times & \times & \times \\ \alpha_1 & \times & \times & \times \\ & \alpha_2 & \times & \times \\ & & \alpha_3 & \times \end{bmatrix} \qquad B = \begin{bmatrix} \times & \times & \times & \times \\ \beta_1 & \times & \times & \times \\ & \beta_2 & \times & \times \\ & & \beta_3 & \times \end{bmatrix}$$

$$\textit{Poles:} \quad \alpha_1/\beta_1 \quad \alpha_2/\beta_2 \quad \alpha_3/\beta_3$$

This is related to the rational Arnoldi algorithm.

Pole Swapping

Poles of a Hessenberg pair

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Hessenberg-triangular form: special case with all poles ∞.

Pole Swapping

Poles of a Hessenberg pair

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This is related to the rational Arnoldi algorithm.

Hessenberg-triangular form: special case with all poles ∞.

Pole-swapping algorithms manipulate the poles.

Pole Swapping

Two types of Moves:

Pole Swapping

Two types of Moves:

I: Change the top (or bottom) pole to any desired value.

Pole Swapping

Two types of Moves:

I: Change the top (or bottom) pole to any desired value.

II: Interchange any two adjacent poles.

Pole Swapping

Move of type I:

introducing a new pole at top

Pole Swapping

Move of type I:

introducing a new pole at top

Pole Swapping

Move of type I:

introducing a new pole at top

Pole Swapping

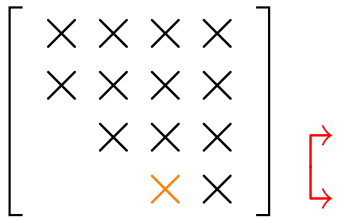
Move of type I:

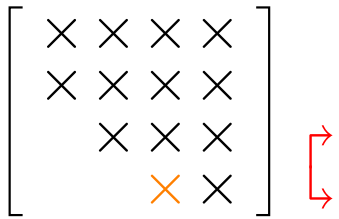
introducing a new pole at bottom

Pole Swapping

Move of type I:

introducing a new pole at bottom





Pole Swapping

Move of type I:

introducing a new pole at bottom

Pole Swapping

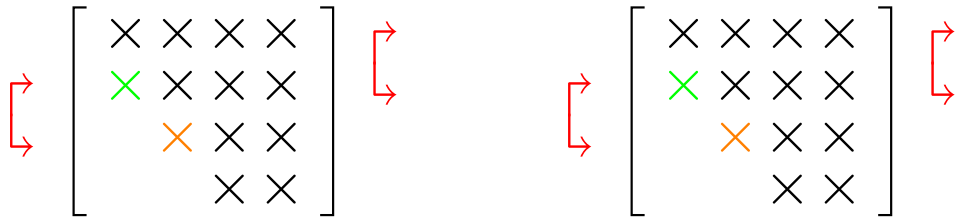
Move of type II:

swapping two adjacent poles

Pole Swapping

Move of type II:

swapping two adjacent poles



Pole Swapping

Move of type II:

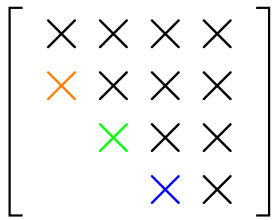
swapping two adjacent poles

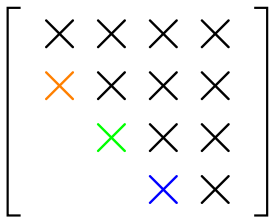
Pole Swapping

A pole swap is an eigenvalue swap!

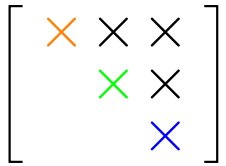
Pole Swapping

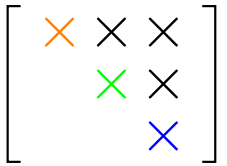
A pole swap is an eigenvalue swap!





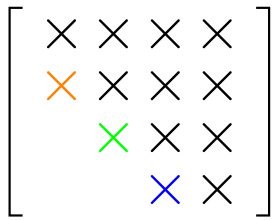
Delete first row and last column to get the *pole pencil*:

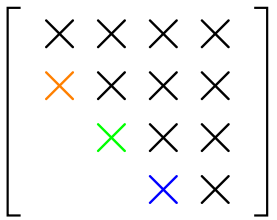




Pole Swapping

A pole swap is an eigenvalue swap!





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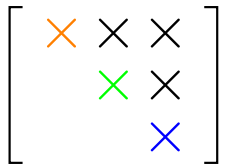
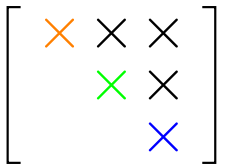
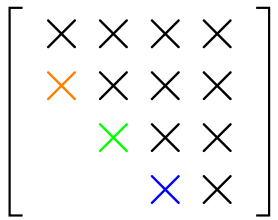


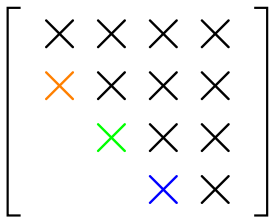


Poles of original pencil are *eigenvalues* of pole pencil.

Pole Swapping

A pole swap is an eigenvalue swap!





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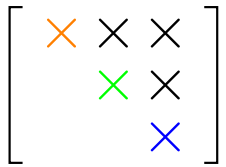
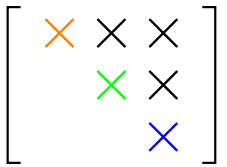


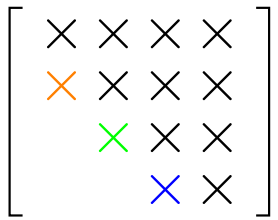


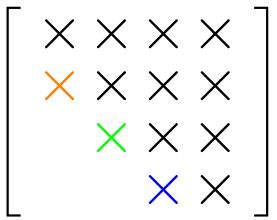
Poles of original pencil are *eigenvalues* of pole pencil.

Swapping poles in original pencil is the same as swapping eigenvalues in pole pencil.

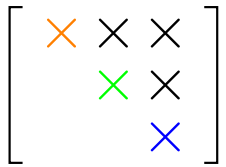
Pole Swapping

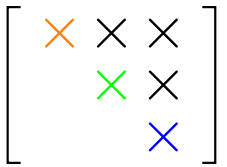
A pole swap is an eigenvalue swap!





Delete first row and last column to get the *pole pencil*:





Poles of original pencil are *eigenvalues* of pole pencil.

Swapping poles in original pencil is the same as swapping eigenvalues in pole pencil. This has been studied.

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Numerous algorithms can be constructed from these two moves.

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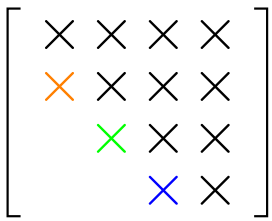
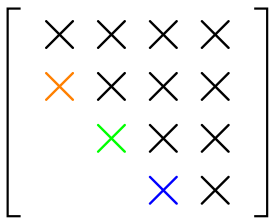
- Remove the shift from the bottom, inserting any desired pole. (type I)

Pole Swapping





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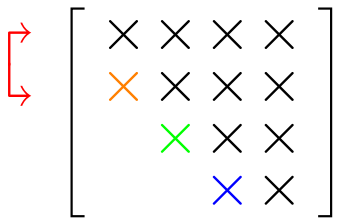

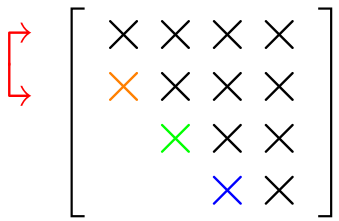

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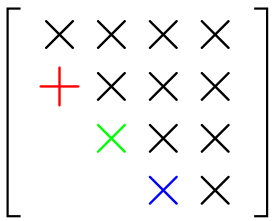

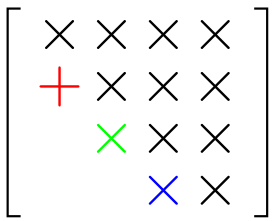

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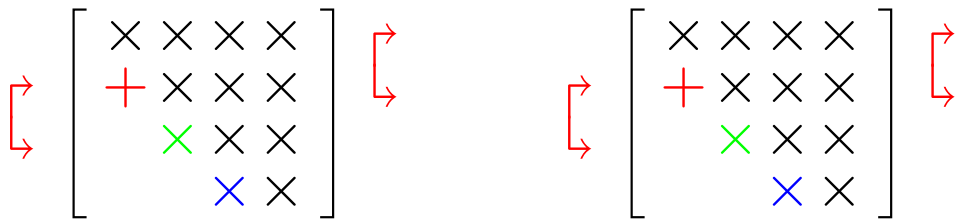

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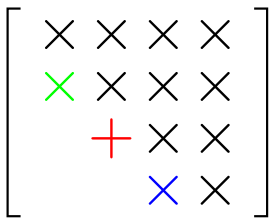

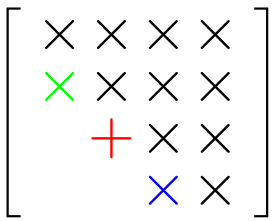

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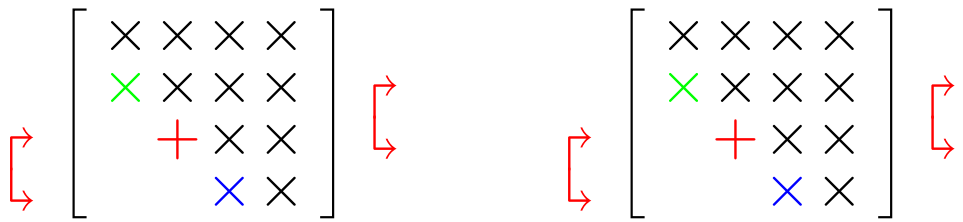

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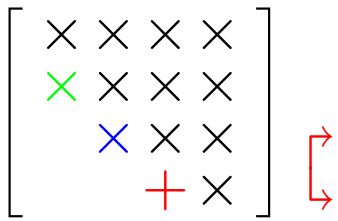

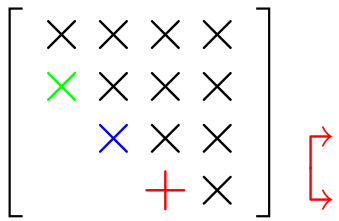

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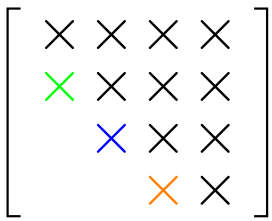

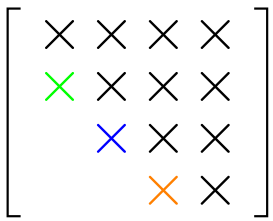

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- Chase shifts in both directions at once.

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We've focused on generalized eigenvalue problem

$$Ax = \lambda Bx,$$

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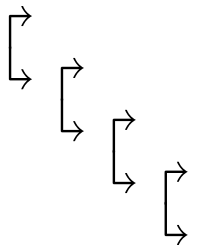
$U =$ 

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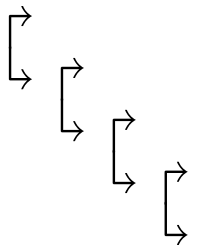
$U =$ 

The resulting algorithm combines pole swapping with core chasing.

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Thank you for your attention!