

# Francis's Algorithm as a Core-Chasing Algorithm

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Department of Mathematics  
Washington State University

PNWNAS, November 12, 2016

# Today's Topic

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- $A \in \mathbb{C}^{n \times n}$

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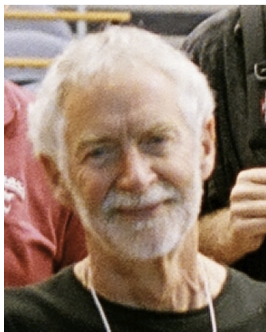
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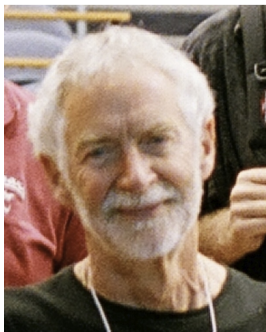
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- Interest dates back to the very beginning of the electronic computing era.

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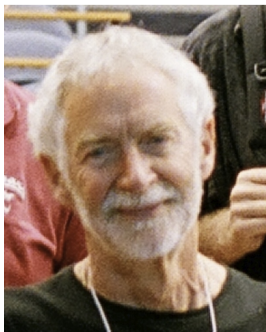
- The matrix eigenvalue problem
- $A \in \mathbb{C}^{n \times n}$
- Find the eigenvalues ( . . . vectors, invariant subspaces)
- Many applications
- Interest dates back to the very beginning of the electronic computing era.
- Nobody knew how to do it.



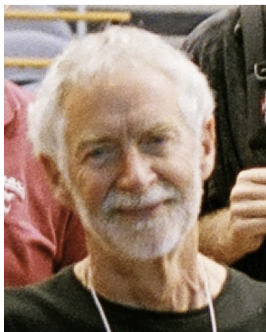




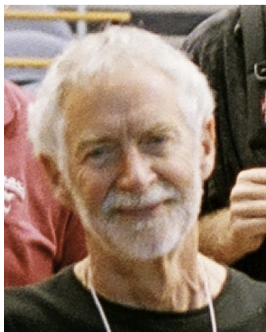
- invented the winning algorithm in 1959.



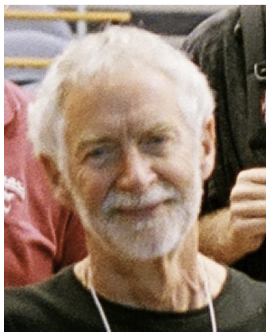
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- better yet: Francis's algorithm, bulge-chasing algorithm.

# Francis's algorithm (superficial description)

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- upper Hessenberg form

$$A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix}$$

- unitary similarity transformation
- direct method ( $O(n^3)$  flops)



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- Drive toward triangular form.

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- unitary similarity transformation
- direct method ( $O(n^3)$  flops)
- Francis: Iterate
- Drive toward triangular form.
- (Galois theory)

# Francis's algorithm (superficial description)

Chasing the bulge

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix}$$

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Chasing the bulge

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Chasing the bulge

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix}$$

- iteration complete!

# Francis's algorithm (superficial description)

Chasing the bulge

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix}$$

- iteration complete!
- repeated iterations  $\Rightarrow$  triangular form

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Chasing the bulge

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix}$$

- iteration complete!
- repeated iterations  $\Rightarrow$  triangular form
- This is the *single-shift* algorithm.

# Francis's algorithm (superficial description)

Chasing the bulge

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix}$$

- iteration complete!
- repeated iterations  $\Rightarrow$  triangular form
- This is the *single-shift* algorithm.
- *Double-shift* algorithm chases a bigger bulge.

## Computational Cost

- $O(n^2)$  flops per iteration
- $O(n)$  total iterations
- $O(n^3)$  total flops

For details see ...

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- Golub and Van Loan, *Matrix Computations*, 4th Ed.

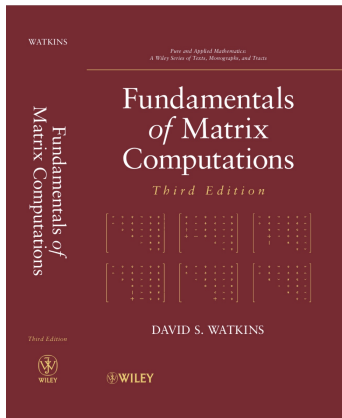
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... but we're still not done!

This is joint work with

- **Jared Aurentz** (Oxford)
- **Thomas Mach** (KU Leuven)
- **Raf Vandebril** (KU Leuven)

# A new look at an old algorithm



# A new look at an old algorithm

Store in QR decomposed form

$$A = QR$$

- $Q$  is unitary,  $R$  is upper triangular

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- looks inefficient!

# A new look at an old algorithm

Store in QR decomposed form


$$A = QR$$

- $Q$  is unitary,  $R$  is upper triangular
- looks inefficient! but it's not!

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$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix}$$

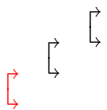
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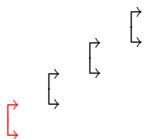
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$$\left[ \begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{array} \right] = \left[ \begin{array}{ccccc} * & * & * & * & * \\ 0 & * & * & * & * \\ & 0 & * & * & * \\ & & * & * & * \\ & & & * & * \end{array} \right]$$

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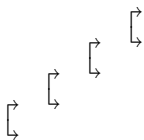

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- **Def:** *Core Transformation*

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- **Def:** *Core Transformation*
- Now invert the core transformations to move them to the other side.

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$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix} = \begin{matrix} \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \end{matrix} \begin{bmatrix} * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \\ & & & & * \end{bmatrix}$$

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$$A = QR$$



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## Manipulating core transformations

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- Fusion

$$\left[ \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right] \left[ \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right] \Rightarrow \left[ \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right]$$



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- Turnover (aka shift through, Givens swap, ...)

$$\left[ \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right] \left[ \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right] \left[ \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right] \Leftrightarrow \left[ \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right] \left[ \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right] \left[ \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \right]$$

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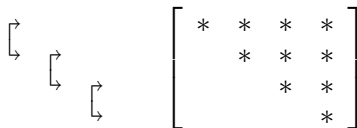
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- Passing a core transformation through a triangular matrix (cost  $O(n)$ )

$$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix} \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \Leftrightarrow \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & * & * \end{bmatrix} \begin{array}{c} \rightarrow \\ \leftarrow \end{array} \Leftrightarrow \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$$

# A new look at an old algorithm

## Francis's algorithm on the QR decomposed form (a core chasing algorithm)

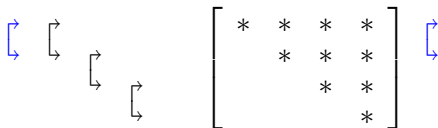


The diagram illustrates the core chasing process in Francis's algorithm. On the left, three horizontal arrows point to the right, with a vertical double-headed arrow bracket positioned above each arrow, indicating the movement of the core across the matrix. On the right, a matrix is shown with asterisks representing non-zero elements. The matrix is upper triangular, with non-zero elements in the first four rows and columns, forming a core of size 4x4. The asterisks are arranged as follows: Row 1 has four asterisks; Row 2 has three asterisks starting from the second column; Row 3 has two asterisks starting from the third column; Row 4 has one asterisk in the fourth column.

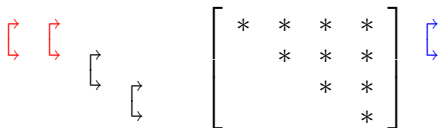
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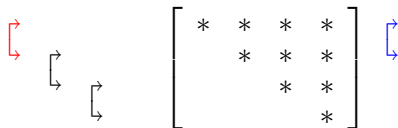


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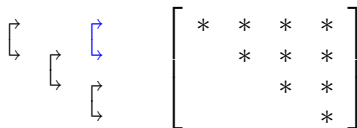


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$$\left[ \begin{array}{cccc} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{array} \right]$$

The diagram illustrates the core-chasing process in Francis's algorithm. It shows a matrix with asterisks representing non-zero elements. A blue double-headed arrow on the right indicates the core being chased.

## Francis's algorithm on the QR decomposed form (a core chasing algorithm)

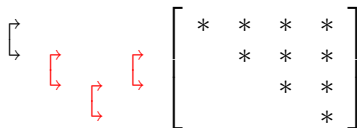
The diagram illustrates the core-chasing process in Francis's algorithm. It shows a matrix in a compact form with asterisks representing non-zero elements. The matrix is:

$$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$$

Four brackets indicate the application of Givens rotations to chase the core:

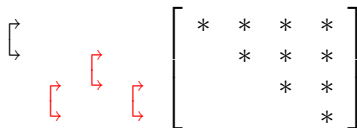
- A white bracket above the first column, spanning rows 1 and 2.
- A white bracket to the left of the second column, spanning rows 2 and 3.
- A white bracket to the left of the third column, spanning rows 3 and 4.
- A blue bracket to the left of the fourth column, spanning rows 4 and 5.

## Francis's algorithm on the QR decomposed form (a core chasing algorithm)

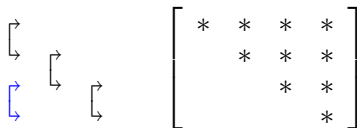

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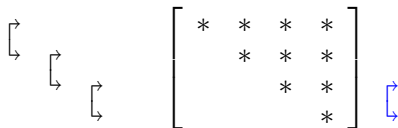
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# A new look at an old algorithm

## Francis's algorithm on the QR decomposed form (a core chasing algorithm)



The diagram illustrates Francis's algorithm on the QR decomposed form. It shows a sequence of three horizontal double-headed arrows pointing to the right, indicating the movement of the core. To the right of these arrows is a matrix structure represented by a large square bracket containing asterisks. The asterisks are arranged in a staircase pattern: the first row has four asterisks, the second row has two asterisks starting from the second column, the third row has two asterisks starting from the third column, and the fourth row has one asterisk in the fourth column. A blue double-headed arrow is positioned to the right of the matrix, pointing to the right, indicating the direction of the core chase.

$$\begin{array}{c} \longleftrightarrow \\ \longleftrightarrow \\ \longleftrightarrow \end{array} \left[ \begin{array}{cccc} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{array} \right] \longleftrightarrow$$

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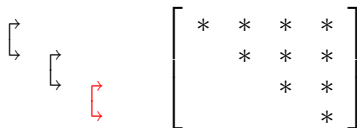
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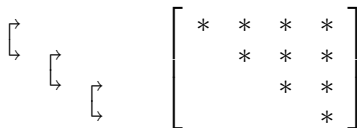
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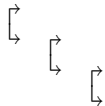


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$$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$$

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## Francis's algorithm on the QR decomposed form (a core chasing algorithm)


$$\begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$$

Done!

## Cost

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- Most arithmetic in passing-through operation



## Cost

- Most arithmetic in passing-through operation
- $O(n^2)$  flops per iteration ...
- $O(n^3)$  total flops ...
- about the same as for standard Francis iteration.

**Are there any advantages?**

## Are there any advantages?

- unitary case

## Are there any advantages?

- unitary case
- companion case (unitary-plus-rank-one)

## Are there any advantages?

- unitary case
- companion case (unitary-plus-rank-one)
- general case: efficient cache use

# Unitary Case

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- Gragg (1986)
- Ammar, Reichel, M. Stewart, Bunse-Gerstner, Elsner, He, W,  
...

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- MATLAB's `roots` command

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  - several methods proposed

# Some of the Competitors

- Chandrasekaran, Gu, Xia, Zhu (2007)
- Bini, Boito, Eidelman, Gemignani, Gohberg (2010)
- Boito, Eidelman, Gemignani, Gohberg (2012)

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- Fortran codes available
- evidence of backward stability
- quasiseparable generator representation
- We will do something else.
- Our method is faster, and **we can prove backward stability.**

# Structure

- Companion matrix is unitary-plus-rank-one

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & -a_0 - 1 \\ 0 & & 0 & -a_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -a_{n-1} \end{bmatrix}$$

- We exploit this structure.

... but we store the QR decomposed form

$$A = QR$$



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$$= \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & -a_1 \\ & 1 & & -a_2 \\ & & \ddots & \vdots \\ & & & -a_0 \end{bmatrix}$$

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# Representation of $R$

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- Next step: Roll up  $x$ .

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$$C_1 \cdots C_{n-1} C_n x = \alpha e_1 \quad (\text{w.l.g. } \alpha = 1)$$



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- $R = P^T C^*(B + e_1 y^T)P = P^T C_n^* \cdots C_1^*(B_1 \cdots B_n + e_1 y^T)P$



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- $R = P^T C^*(B + e_1 y^T)P = P^T C_n^* \cdots C_1^*(B_1 \cdots B_n + e_1 y^T)P$
- $O(n)$  storage
- Bonus: Redundancy! No need to keep track of  $y$ .

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Altogether we have

$$A = QR$$

$$= \begin{matrix} \begin{matrix} \left[ \begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} \right] \\ \left[ \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} \right] \\ \left[ \begin{matrix} \rightarrow \end{matrix} \right] \end{matrix} & \begin{matrix} \left[ \begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} \right] \\ \left[ \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} \right] \\ \left[ \begin{matrix} \rightarrow \end{matrix} \right] \end{matrix} & \left[ \begin{matrix} \left[ \begin{matrix} \rightarrow \\ \rightarrow \\ \rightarrow \end{matrix} \right] \\ \left[ \begin{matrix} \rightarrow \\ \rightarrow \end{matrix} \right] \\ \left[ \begin{matrix} \rightarrow \end{matrix} \right] \\ + \dots \end{matrix} \right] \end{matrix}$$

# Francis Iterations

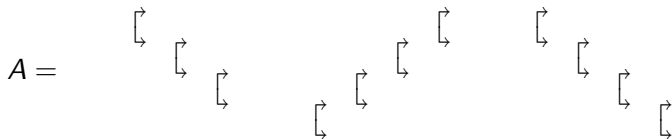
- We have complex single-shift code ...
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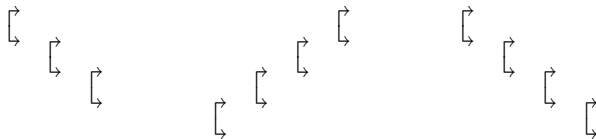
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- ignoring rank-one part ...

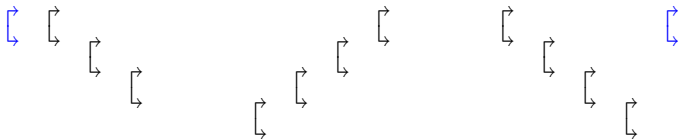


# The Core Chase

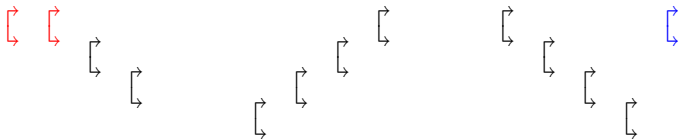




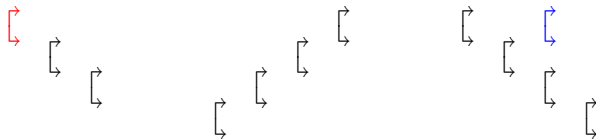
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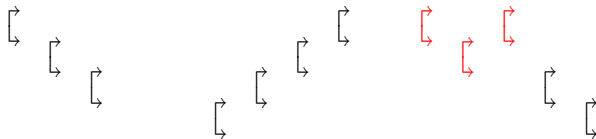
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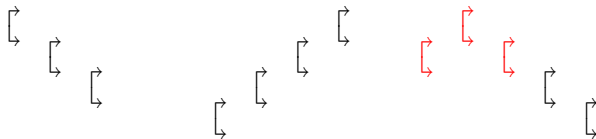
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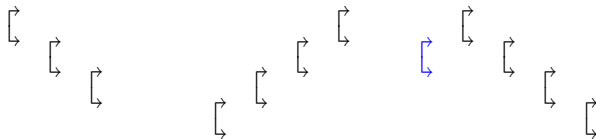
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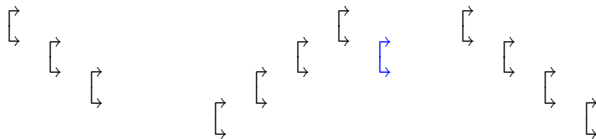
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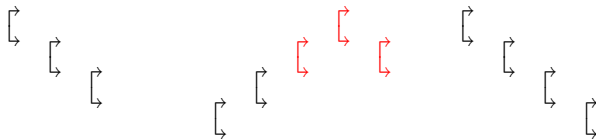
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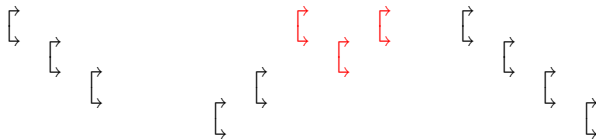


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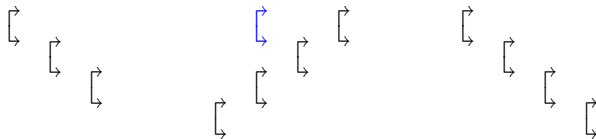




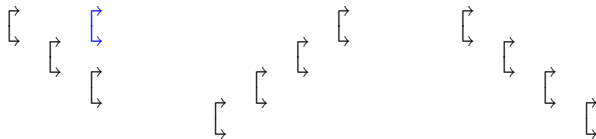
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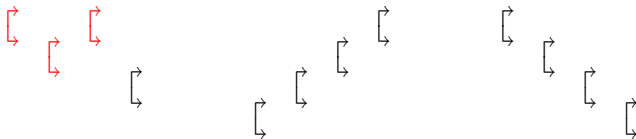
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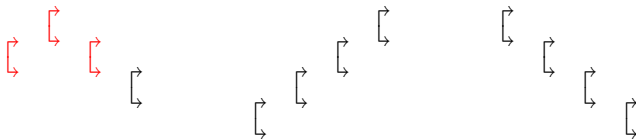
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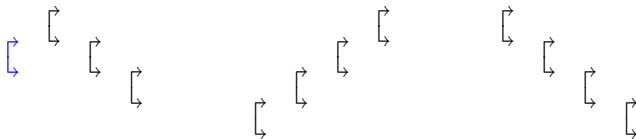
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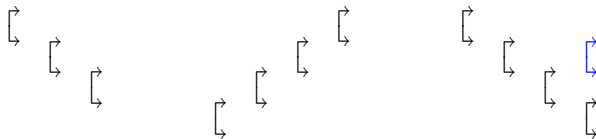
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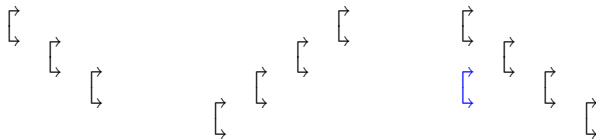
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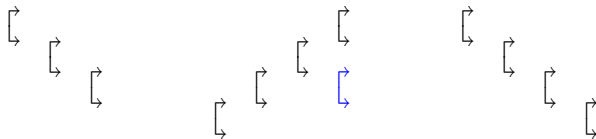


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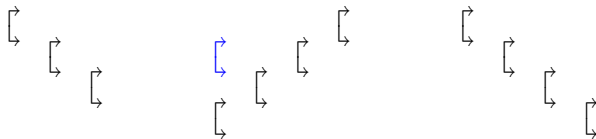




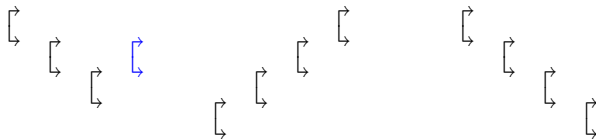
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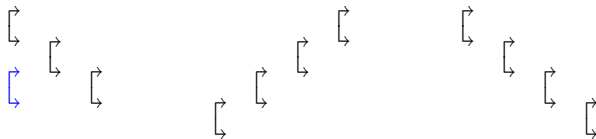
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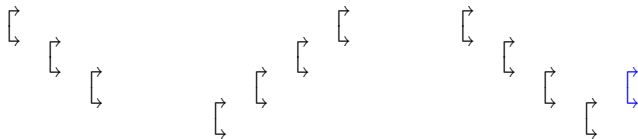
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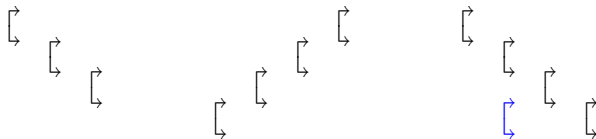
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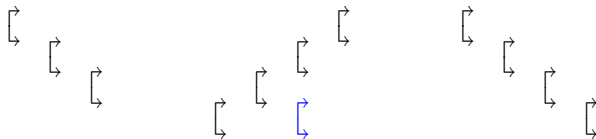
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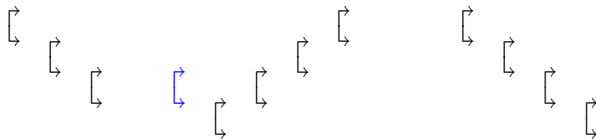
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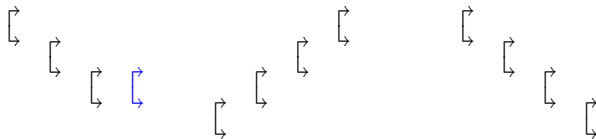


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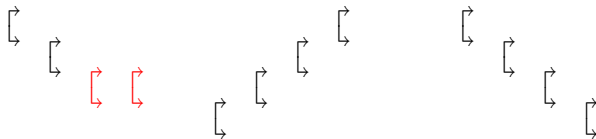




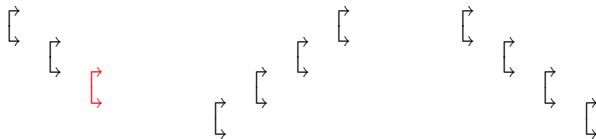
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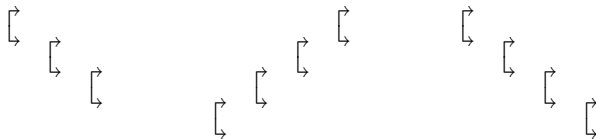
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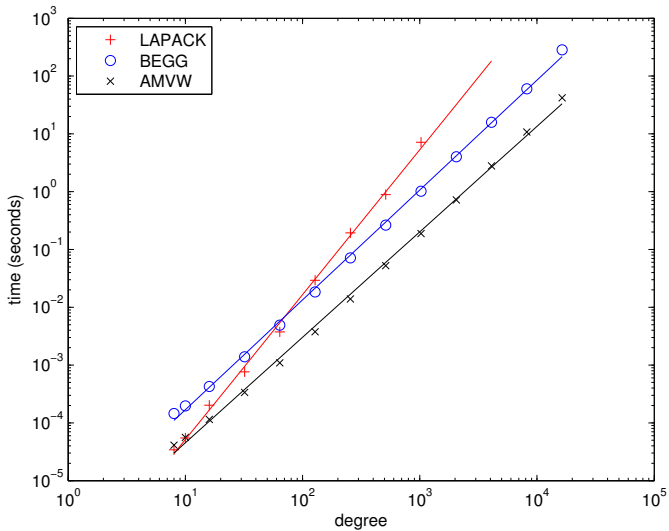
# Done!

- iteration complete!

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- Cost:  $3n$  turnovers/iteration, so  $O(n)$  flops/iteration

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- Double-shift iteration is similar.
- (Chase two core transformations instead of one.)

# Performance





**At degree 1000**

method	time
LAPACK	7.2
BEGG	1.2
AMVW	0.2

# See our paper for . . .

- Paper in SIAM J. Matrix Anal. Appl. has

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- . . . more timings,

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- . . . more timings,
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- Paper in SIAM J. Matrix Anal. Appl. has
- . . . more timings,
- . . . accuracy comparisons,
- . . . proof of backward stability.

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Thank you for your attention.