Core-Chasing Algorithms for the Eigenvalue Problem

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Washington State University

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Our International Research Group

This is joint work with

- **Jared Aurentz** (Oxford $\rightarrow$ Madrid)
- **Thomas Mach** (KU Leuven $\rightarrow$ Astana)
- **Raf Vandebril** (KU Leuven)

Today's Topic

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- The matrix eigenvalue problem

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  - symmetric
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  - symmetric
  - symmetric-plus-rank-one (colleague)
  - . . . or no special structure

John Francis

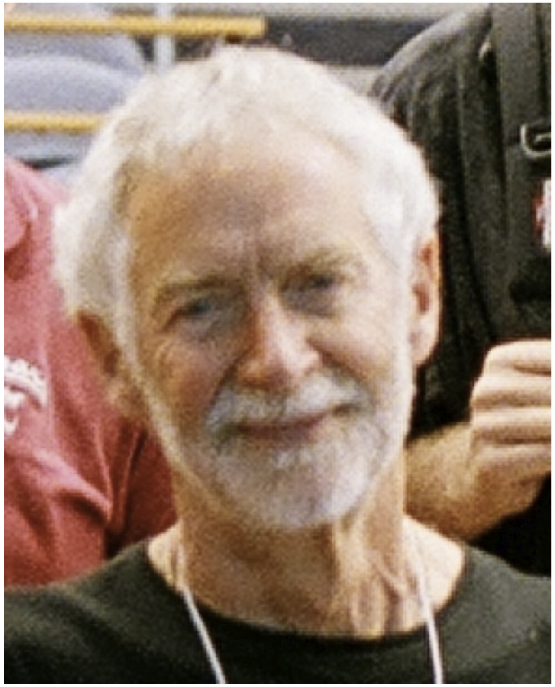

photo: Frank Uhlig, 2009

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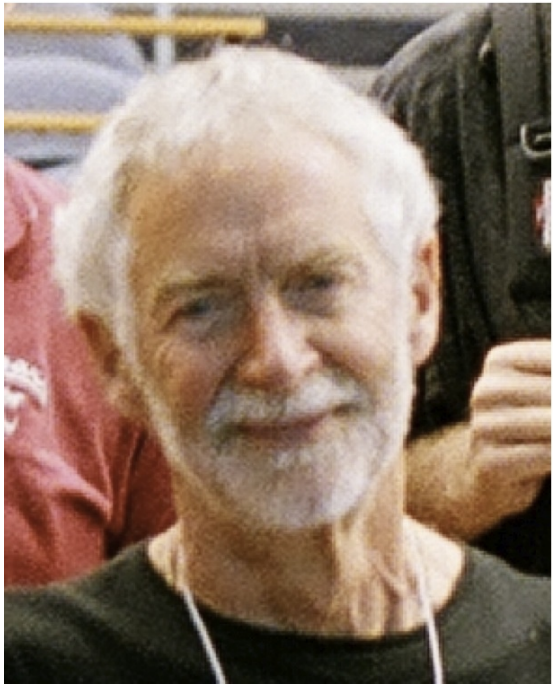

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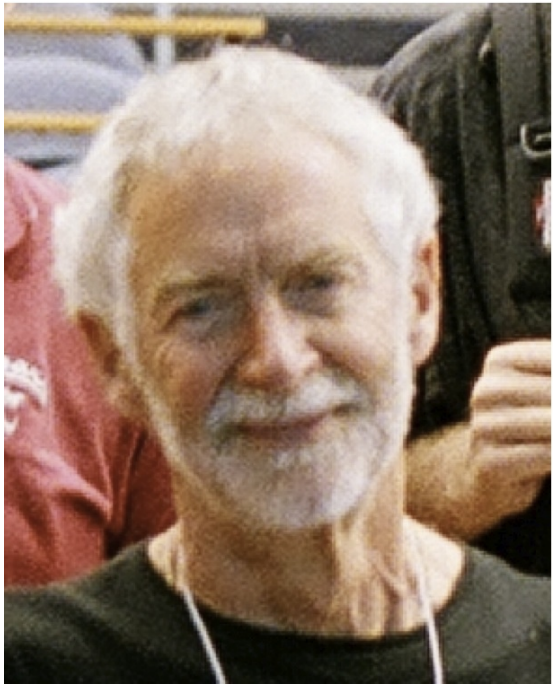

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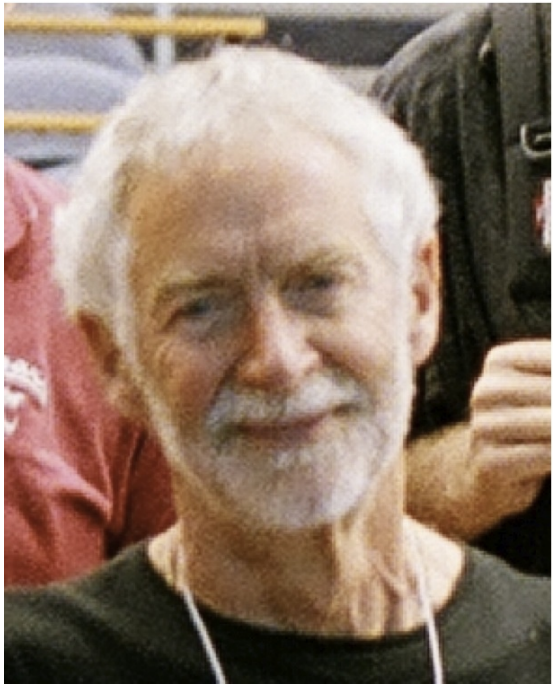

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- invented the winning algorithm in 1959.
- implicitly shifted QR algorithm
- Our algorithms are all variants of this.

- Francis's algorithm . . .

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- Francis's algorithm . . .
- . . . is a bulge chasing algorithm.
- We turn it into a core chasing algorithm.
- Instead of chasing bulges, we chase core transformations.

Core Transformations

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- it's essentially 2×2
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- Ex: Givens rotator, reflector, ...
- We just wanted a generic term.

Core Transformations

$$\begin{bmatrix} 1 & & \\ & \times & \times \\ & \times & \times \end{bmatrix} \begin{bmatrix} \times & \times & \times \\ & \times & \times \\ & \times & \times \end{bmatrix} = \begin{bmatrix} \times & \times & \times \\ & \times & \times \\ & 0 & \times \end{bmatrix}$$

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Abbreviated notation

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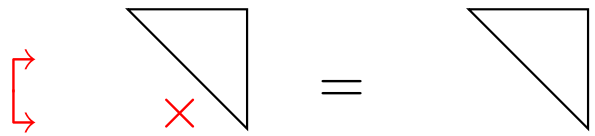
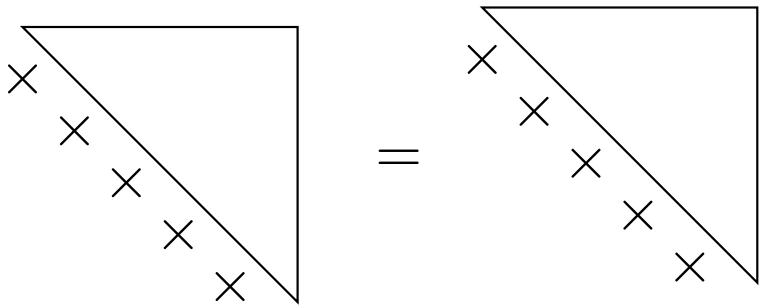


Hessenberg QR decomposition



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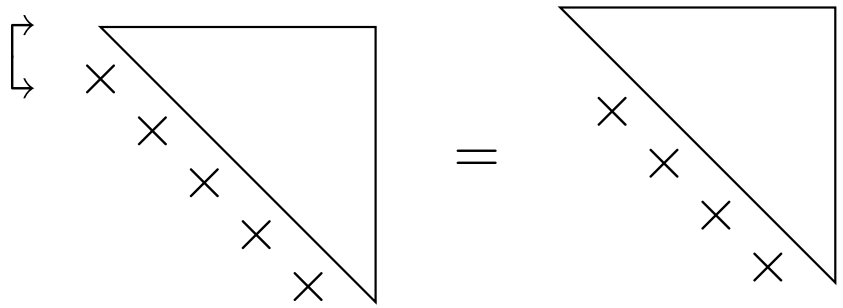

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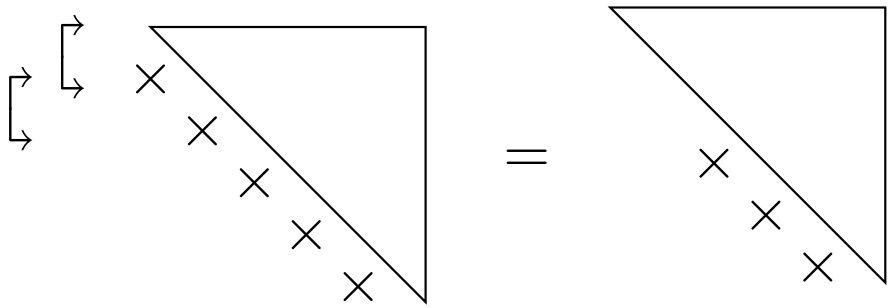

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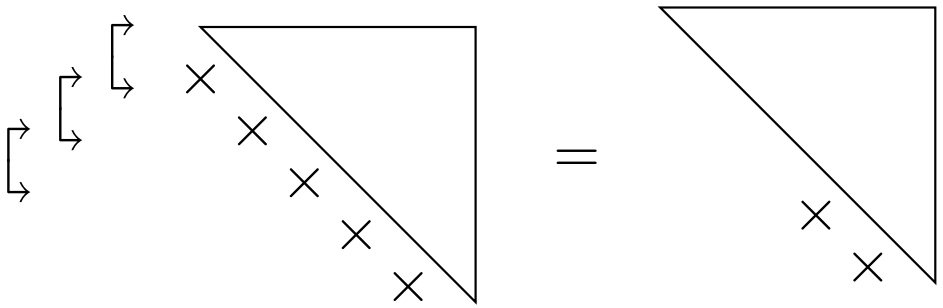

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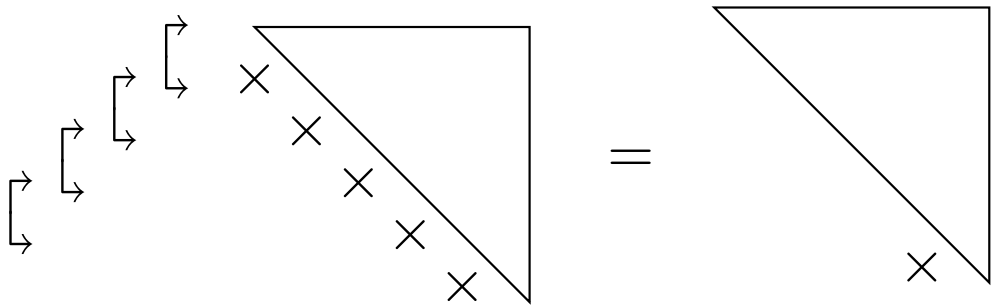

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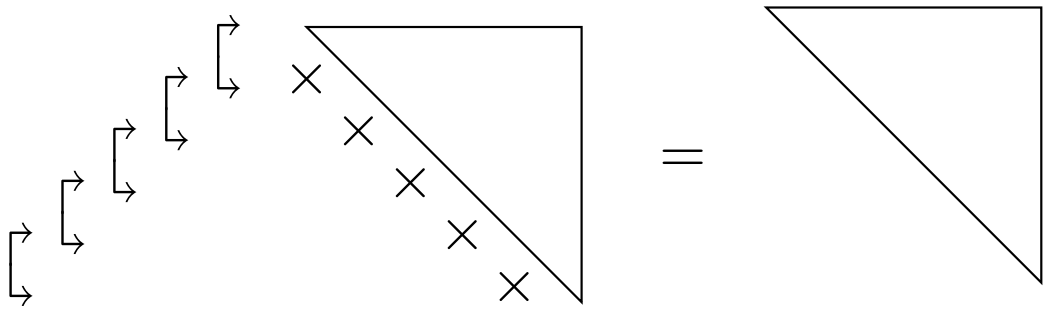

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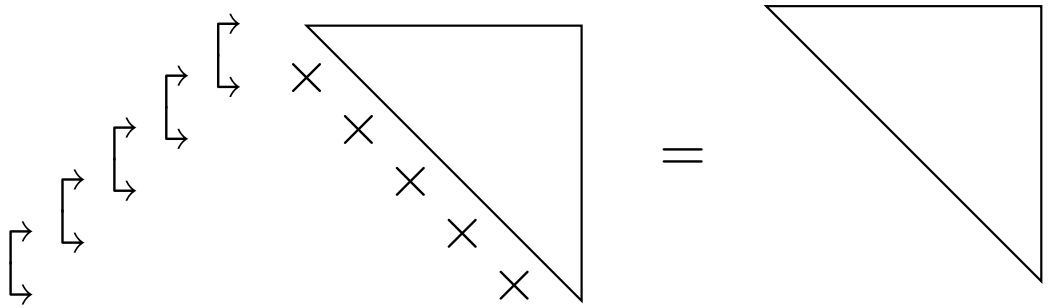
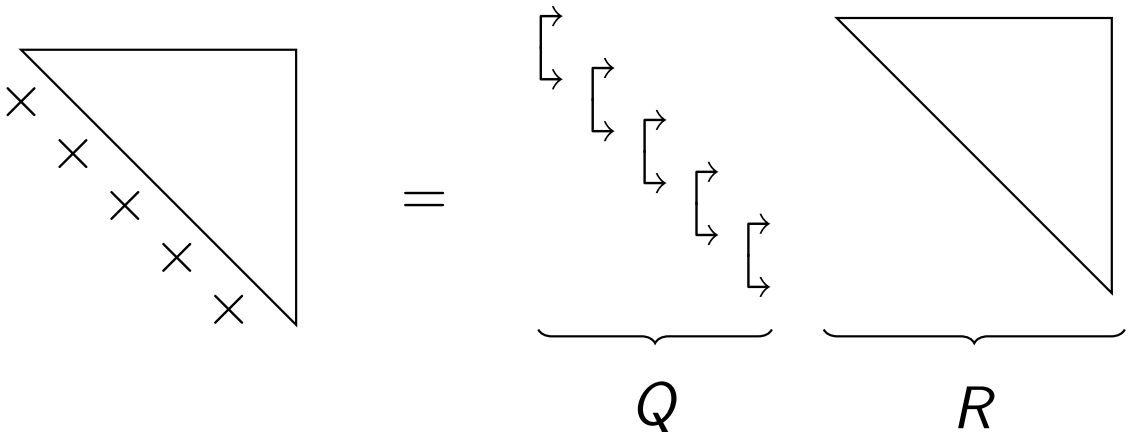


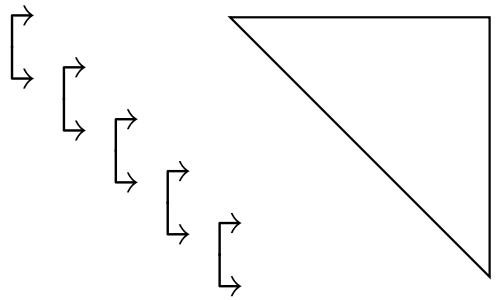
Now invert the core transformations.

Hessenberg QR decomposition



Our algorithms operate on the matrix in QR decomposed form.

$$A = QR =$$

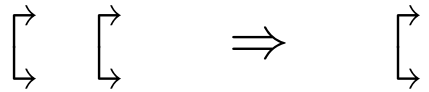


This is not inefficient.

We apply Francis's algorithm to this factored form.

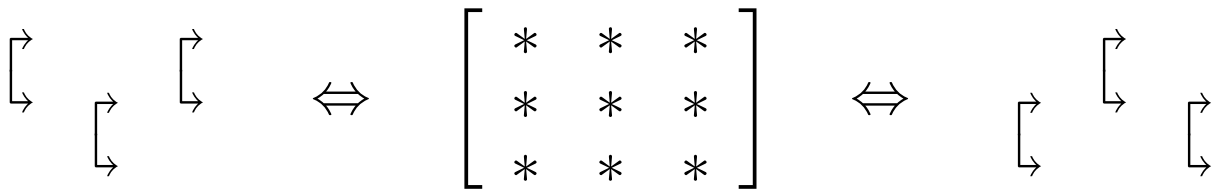
Operating on Core Transformations

Fusion



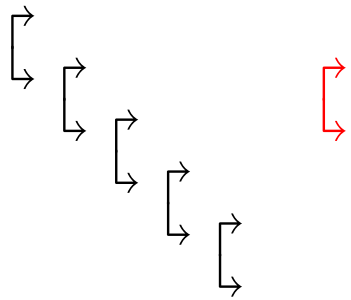
Operating on Core Transformations

Turnover



Operating on Core Transformations

Turnover as a shift-through operation



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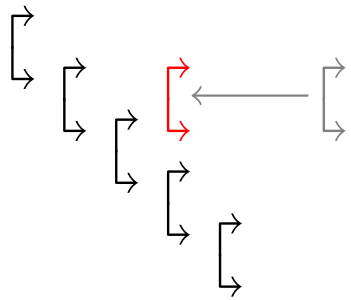

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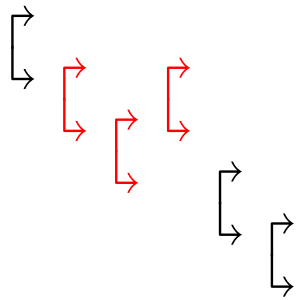

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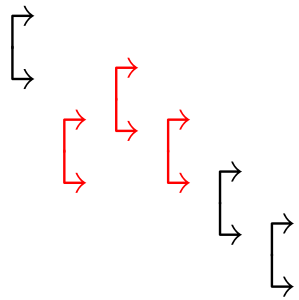

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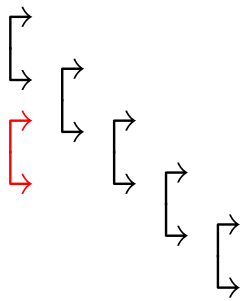

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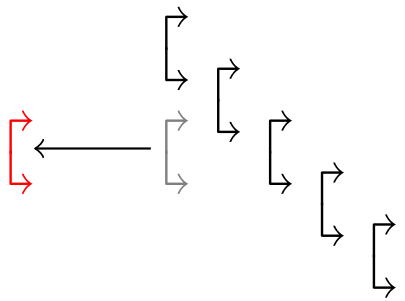

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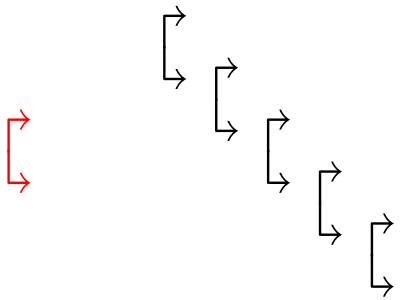
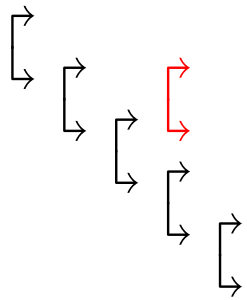


Operating on Core Transformations

Turnover as a shift-through operation : Abbreviated notation



Operating on Core Transformations

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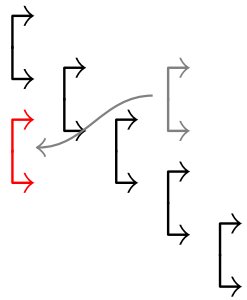
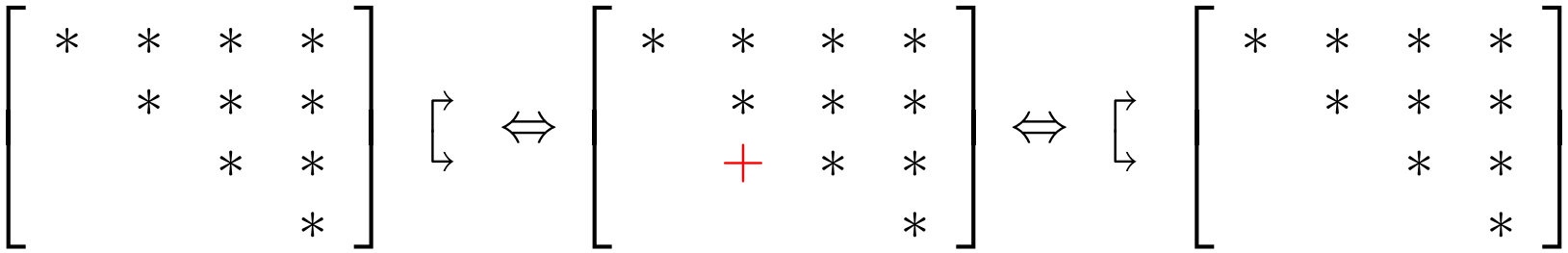


Operating on Core Transformations

Passing a core transformation through a triangular matrix

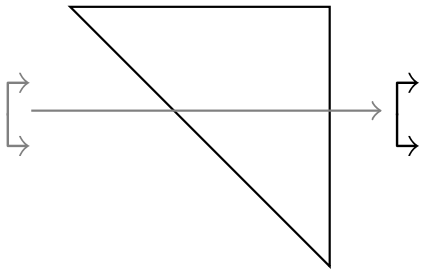


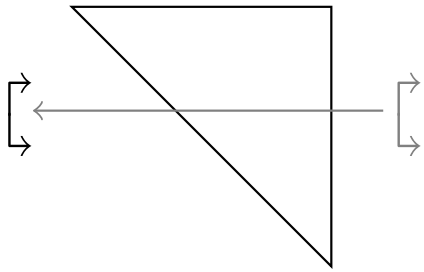
Cost is $O(n)$ flops.

Operating on Core Transformations

Passing a core transformation through a triangular matrix

Abbreviated Notation:





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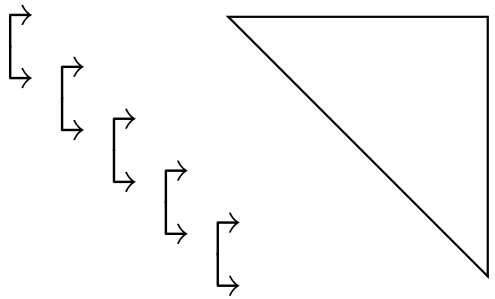

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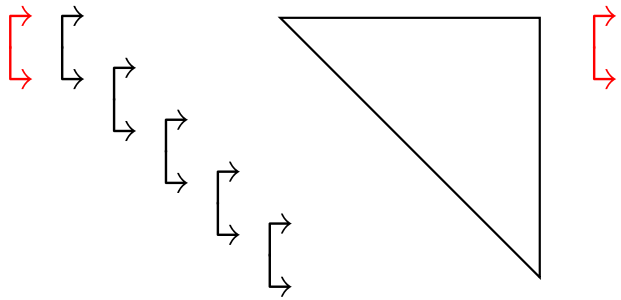

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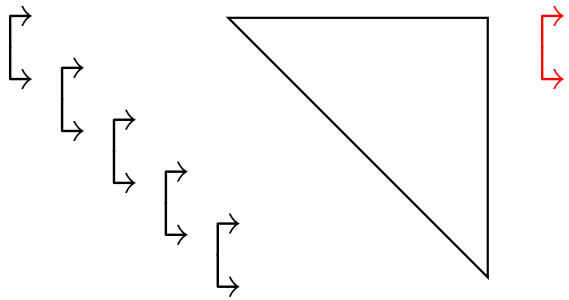

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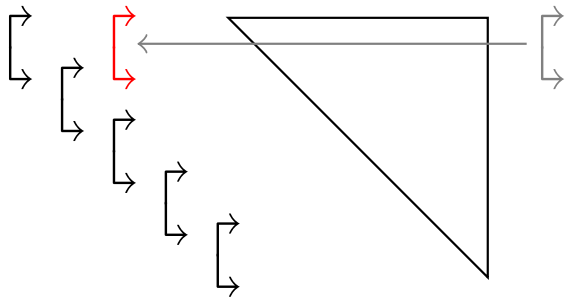

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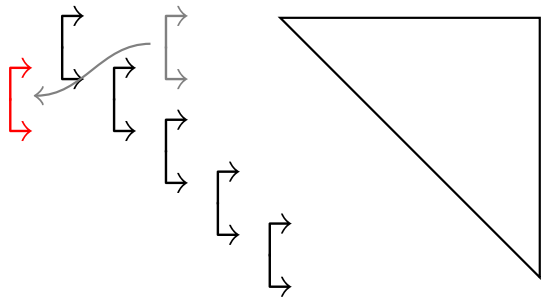

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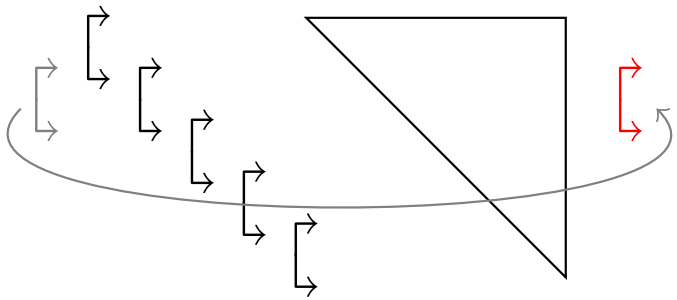

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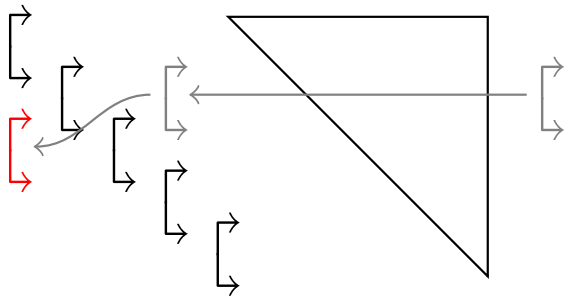

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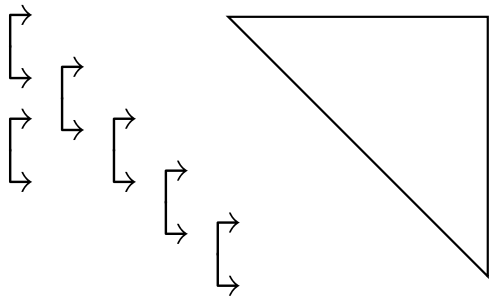

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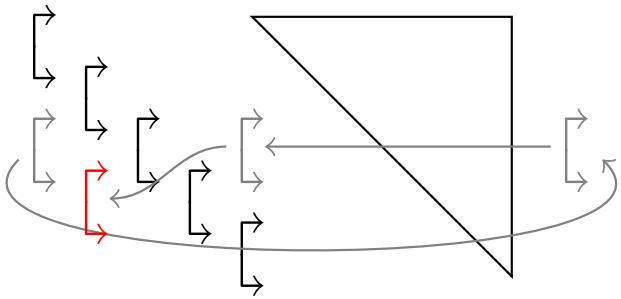

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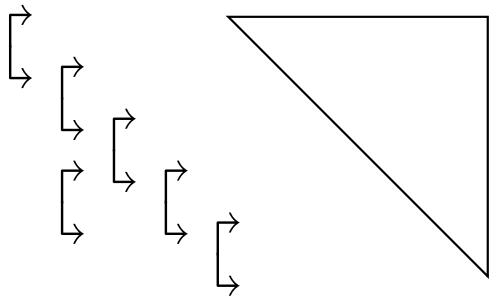

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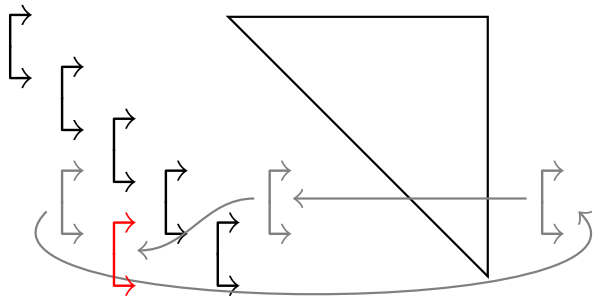

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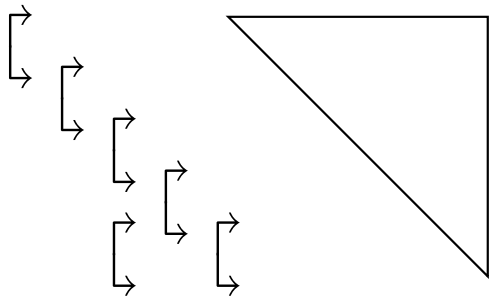

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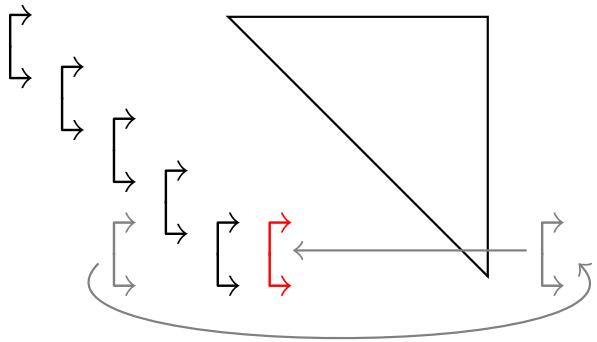

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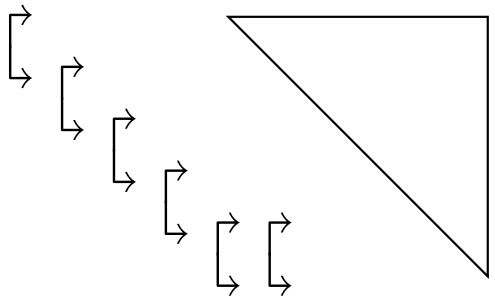

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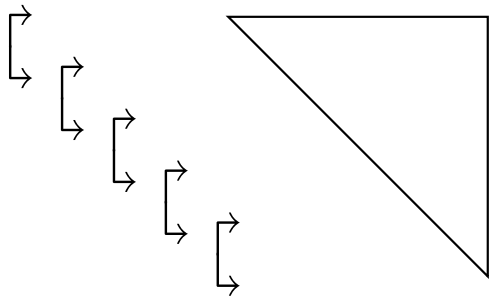


Cost

Cost

- Most arithmetic in passing-through operation

Cost

- Most arithmetic in passing-through operation
- $O(n^2)$ flops per iteration . . .
- $O(n^3)$ total flops . . .
- about the same as for standard Francis iteration.

Are there any advantages?

Are there any advantages?

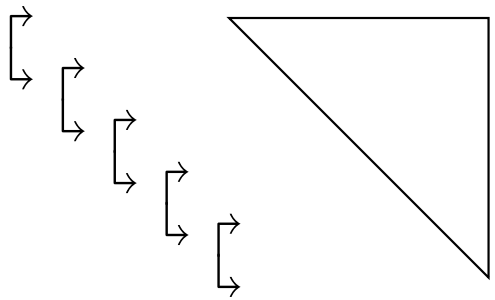
- unitary case

Are there any advantages?

- unitary case
- companion case (unitary-plus-rank-one)

Unitary Case

Unitary Case

$A = QR =$ 

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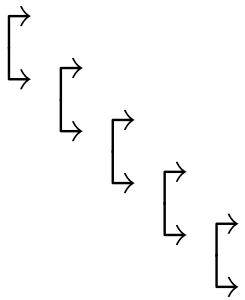
$A = QR =$ 

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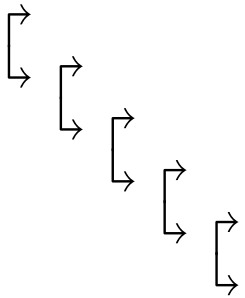
$A = QR =$ 

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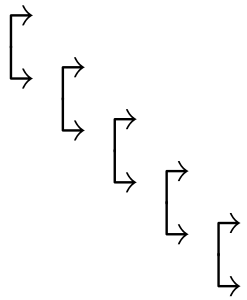
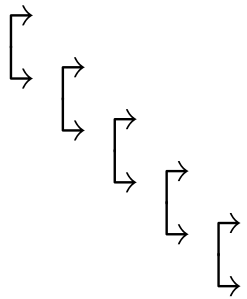
$A = QR =$ 

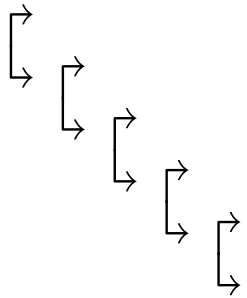
- Cost is $O(n)$ flops per iteration,

Unitary Case

$A = QR =$ 

- Cost is $O(n)$ flops per iteration, $O(n^2)$ flops total.

Unitary Case

$A = QR =$ 

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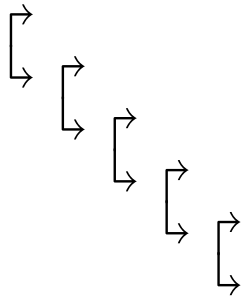
$A = QR =$ 

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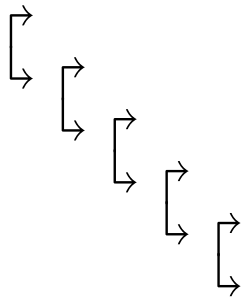
$A = QR =$ 

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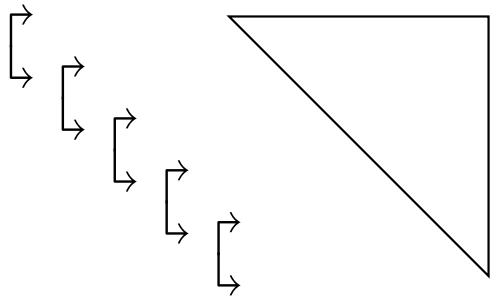
We store the QR decomposed form.

$A = QR =$ 

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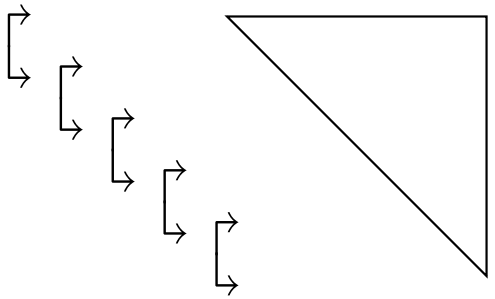
$$A = QR = \quad$$ 

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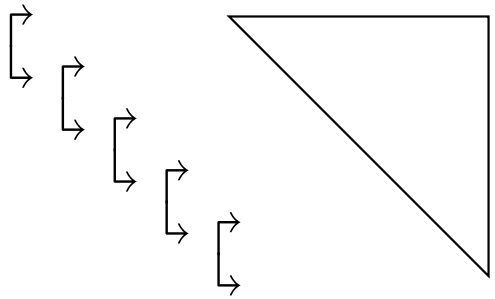
$$A = QR = \quad$$ 

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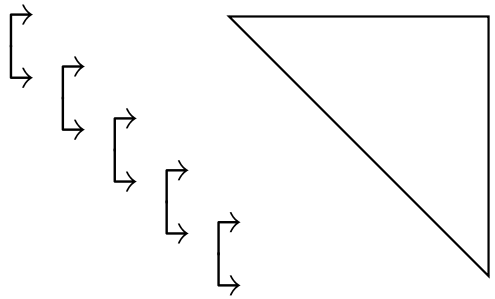
$$A = QR = \quad$$ 

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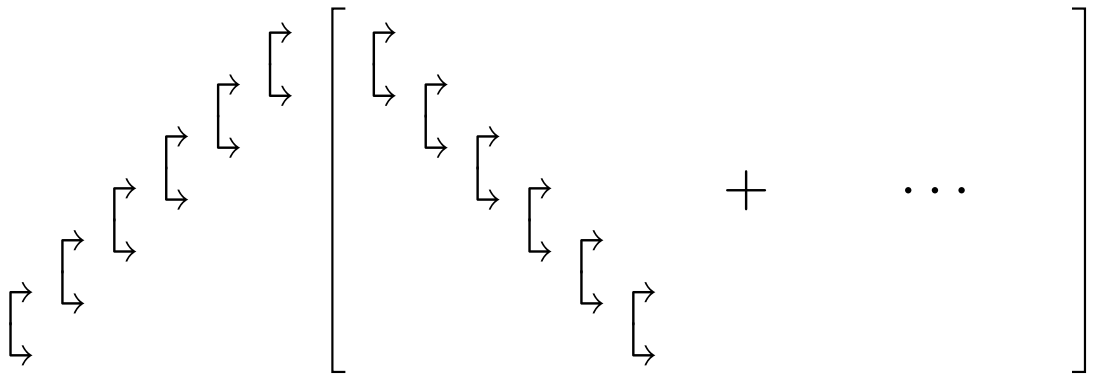

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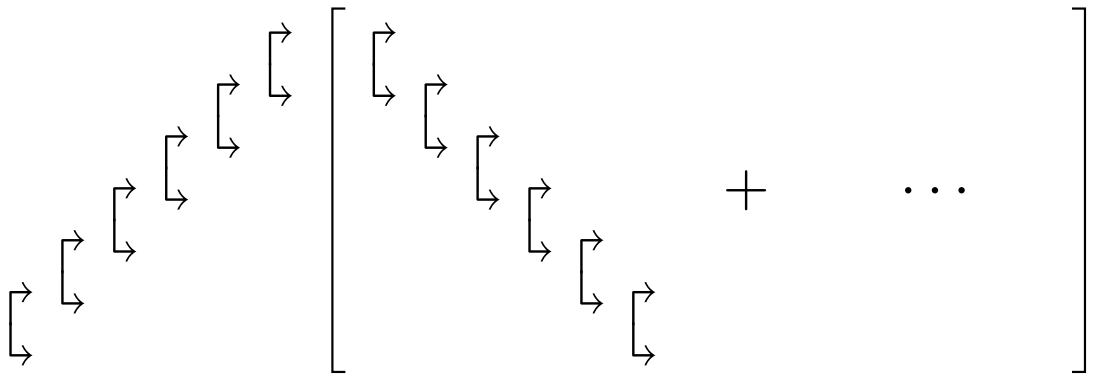

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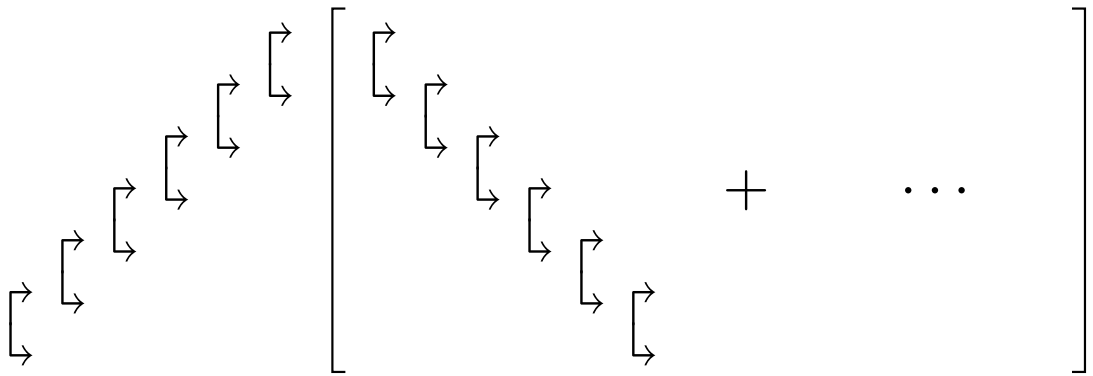

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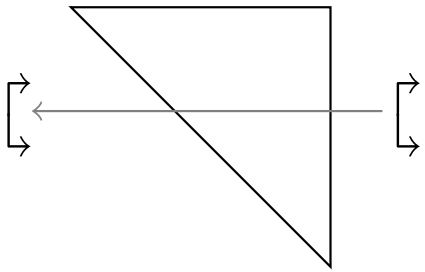
Storage is $O(n)$.

Passing a core transformation through R



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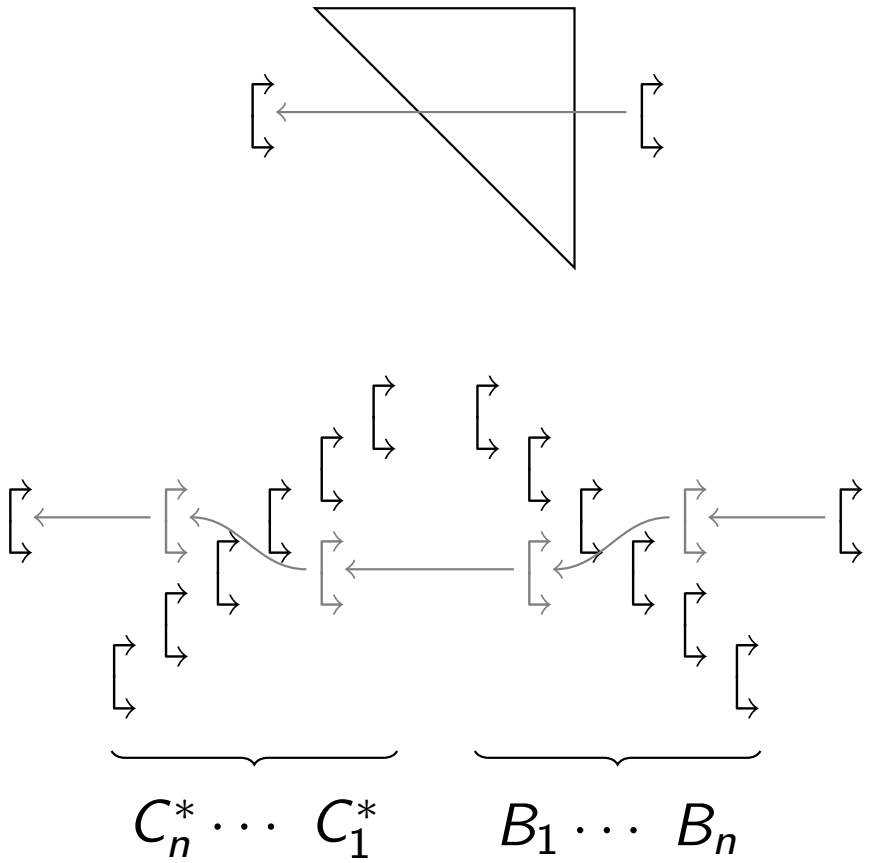

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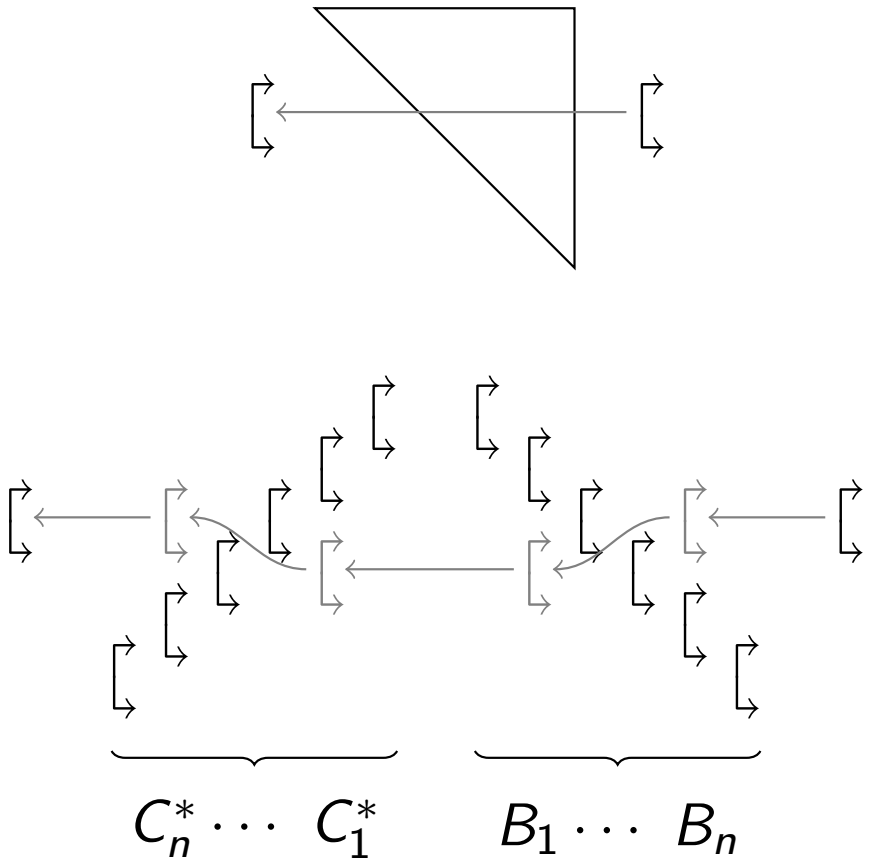

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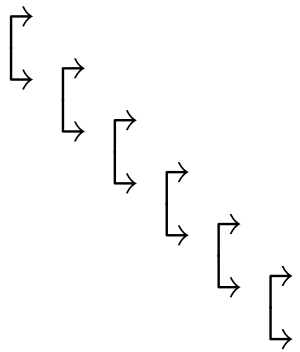
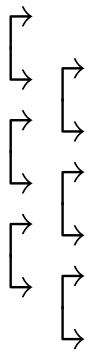
Monograph in progress (100+ pp?)

Beyond Hessenberg form

Form of Q can be upper Hessenberg, so-called CMV, . . .





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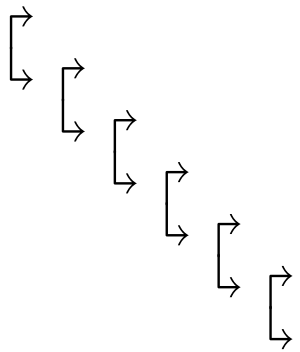
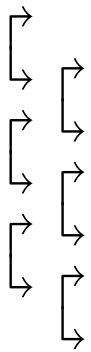
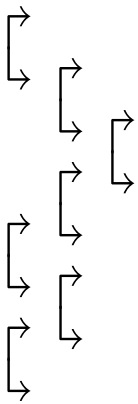




or "random" twisted . . .



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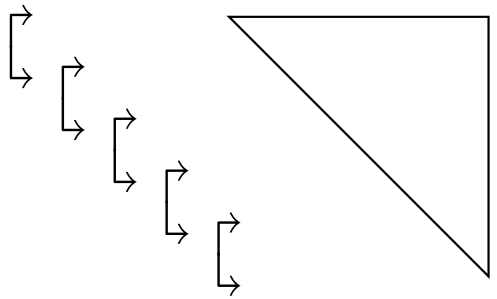
$A = QR =$ 

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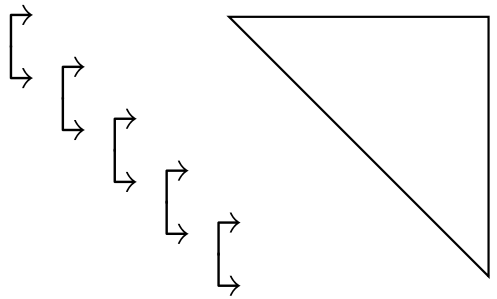
$A = QR =$ 

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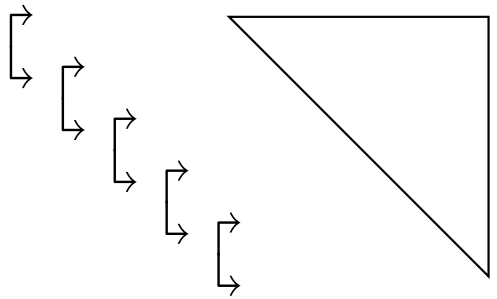
$A = QR =$ 

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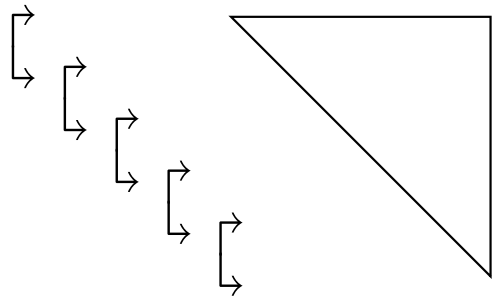
$A = QR =$ 

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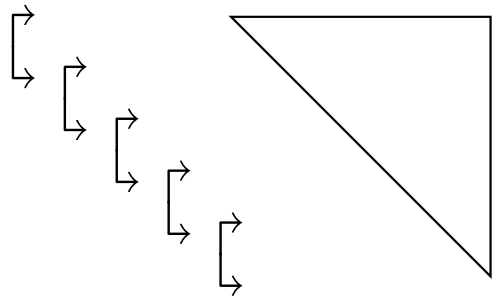
$A = QR =$ 

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$$\| E_q \| \approx u \qquad \| E_r \| \approx u \| R \| = u \| A \|$$

. . . assuming R is stored in the conventional way.

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where

$$E = E_qR + QE_r + O(u^2), \quad \text{so} \quad \|E\| \approx u\|R\|.$$

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$$\hat{C} = W^* C V \qquad \hat{B} = W^* B U$$

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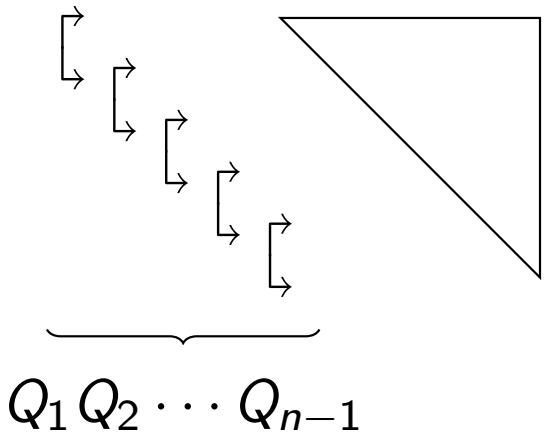

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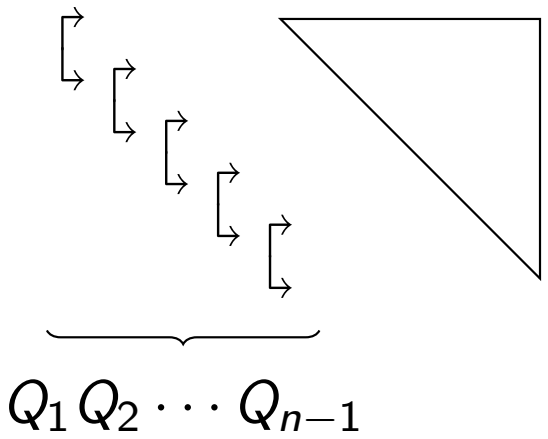

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- Stronger backward stability (Mach and Vandebril 2014)

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Thank you for your attention.