Fast Stable Computation of the Eigenvalues of Companion Matrices

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Collaborators

This is joint work with

- **Jared Aurentz** (WSU)
- **Thomas Mach** (KU Leuven)
- **Raf Vandebril** (KU Leuven)

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The Problem

- $p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 = 0$
- companion matrix

$$A = \begin{bmatrix} 0 & & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{bmatrix}$$

- Get the zeros of p by computing the eigenvalues of A.

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Cost

- If structure not exploited:
  - $O(n^2)$ storage, $O(n^3)$ flops
  - Francis's implicitly-shifted QR algorithm
- If structure exploited:
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  - Several methods proposed
  - including some by us (lightning fast but not stable)

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Some of the Competitors

- Chandrasekaran, Gu, Xia, Zhu (2007)
- Bini, Boito, Eidelman, Gemignani, Gohberg (2010)
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- Fortran codes available
- quasiseparable generator representation
- We will do something else.
- evidence of backward stability

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Our Contribution

We present

- Yet another $O(n)$ representation
- Francis algorithm in $O(n)$ flops/iteration
- Fortran codes (we're faster)
- normwise backward stable (We can prove it.)

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Structure

- Companion matrix is unitary-plus-rank-one

$$\begin{bmatrix} 0 & \cdots & 0 & e^{i\theta} \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & -e^{i\theta} - a_0 \\ 0 & & 0 & -a_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -a_{n-1} \end{bmatrix}$$

- preserved by unitary similarities
- Companion matrix is also upper Hessenberg.
- preserved by Francis algorithm
- We exploit this structure.

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- Chandrasekaran, Gu, Xia, Zhu (2007)
- $A = QR$
- Q is upper Hessenberg and unitary.
- R is upper triangular and unitary-plus-rank-one.
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The Unitary Part

$$\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ & x & x & x \\ & & x & x \end{bmatrix} = \begin{bmatrix} x & x & & \\ x & x & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & x & x & \\ & x & x & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & x & x \\ & & x & x \end{bmatrix}$$

$Q =$

- $O(n)$ storage

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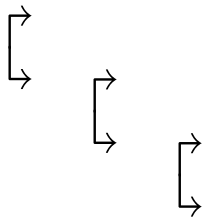
$Q =$ 

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$$Q = \quad \begin{matrix} \cdot & & \\ & \cdot & \\ & & \cdot \end{matrix}$$

- $O(n)$ storage

The Upper Triangular Part

- $R = U + xy^T$ unitary-plus-rank-one, so
- R has quasiseparable rank 2.

$$R = \left[\begin{array}{ccc|ccc} x & \cdots & x & x & \cdots & x \\ & \ddots & \vdots & \vdots & & \vdots \\ & & x & x & \cdots & x \\ \hline & & & x & \cdots & x \\ & & & & \ddots & \vdots \\ & & & & & x \end{array}\right]$$

- quasiseparable generator representation ($O(n)$ storage)
- Chandrasekaran et. al. exploit this structure.
- We do it differently.

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Our Representation

- Add a row/column for extra wiggle room

$$A = \left[\begin{array}{cccc|c} 0 & & & -a_0 & 1 \\ 1 & & & -a_1 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 1 & -a_{n-1} & 0 \\ \hline & & & 0 & 0 \end{array}\right]$$

- Extra zero root can be deflated immediately.
- $A = QR$, where

$$Q = \left[\begin{array}{cccc|c} 0 & & & \pm 1 & 0 \\ 1 & & & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 1 & 0 & 0 \\ \hline & & & 0 & 1 \end{array}\right] \qquad R = \left[\begin{array}{cccc|c} 1 & & & -a_1 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 1 & -a_{n-1} & 0 \\ & & & \pm a_0 & \mp 1 \\ \hline & & & 0 & 0 \end{array}\right]$$

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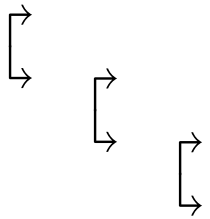
- Q is stored in factored form
- Q $=$ 
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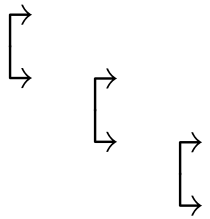
- Q = 
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$$R = \left[\begin{array}{ccc|c} 1 & & & -a_1 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 1 & -a_{n-1} & 0 \\ & & & \pm a_0 & \mp 1 \\ \hline & & & 0 & 0 \end{array}\right]$$

- R is unitary-plus-rank-one:

$$\left[\begin{array}{ccc|c} 1 & & & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 1 & 0 & 0 \\ & & & 0 & \mp 1 \\ \hline & & & \pm 1 & 0 \end{array}\right] + \left[\begin{array}{ccc|c} 0 & & & -a_1 & 0 \\ & \ddots & & \vdots & \vdots \\ & & 0 & -a_{n-1} & 0 \\ & & & \pm a_0 & 0 \\ \hline & & & \mp 1 & 0 \end{array}\right]$$

Representation of R

- $R = U + xy^T$, where

$$xy^T = \left[\begin{array}{c} -a_1 \\ \vdots \\ -a_{n-1} \\ \pm a_0 \\ \hline \mp 1 \end{array}\right] \left[\begin{array}{ccccc|c} 0 & \cdots & 0 & 1 & 0 \end{array}\right]$$

- Next step: Roll up x.

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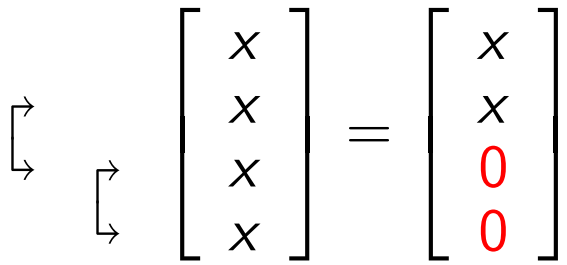

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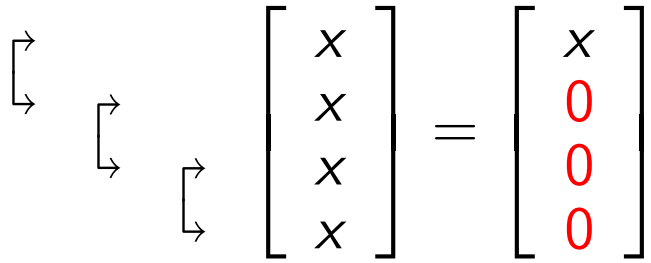

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$$C_1 \cdots C_{n-1} C_n x = \alpha e_1 \qquad \text{(w.l.g. } \alpha = 1\text{)}$$

Representation of R

- $C_1 \cdots C_{n-1} C_n x = e_1$
- $Cx = e_1$
- $C^* e_1 = x$
- $R = U + xy^T = U + C^* e_1 y^T = C^*(CU + e_1 y^T)$
- $R = C^*(B + e_1 y^T)$
- B is upper Hessenberg (and unitary) so $B = B_1 \cdots B_n$.
- $R = C^*(B + e_1 y^T) = C_n^* \cdots C_1^*(B_1 \cdots B_n + e_1 y^T)$
- $O(n)$ storage
- Bonus: Redundancy! No need to keep track of y.

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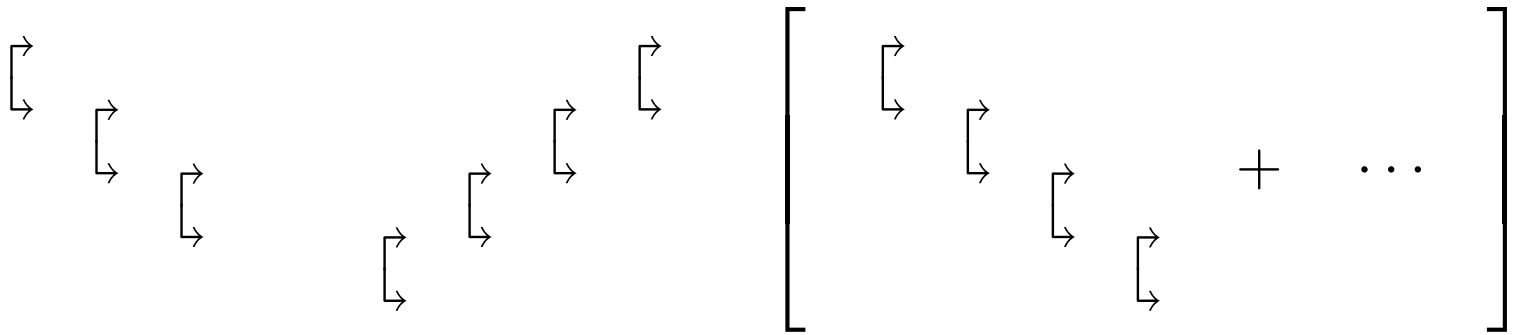
Representation of A

Altogether we have

- $A = QR = Q\ C^* (B + e_1 y^T)$
- $A = Q_1 \cdots Q_{n-1}\, C_n^* \cdots C_1^* (B_1 \cdots B_n + e_1 y^T)$



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- We have complex single-shift code . . .
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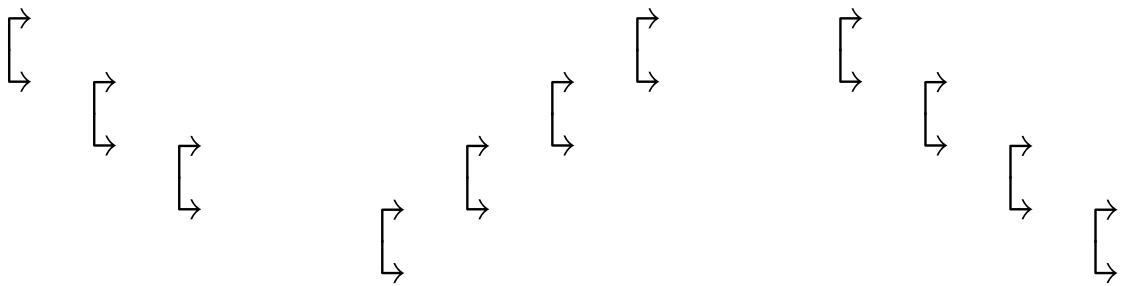
$A =$ 

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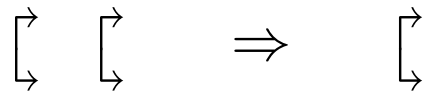

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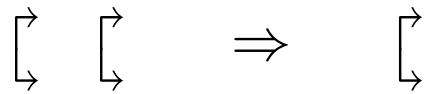

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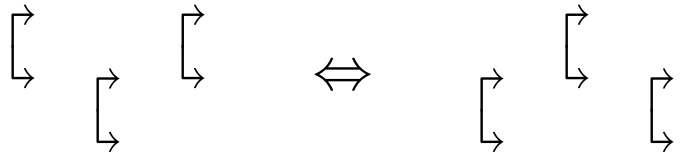
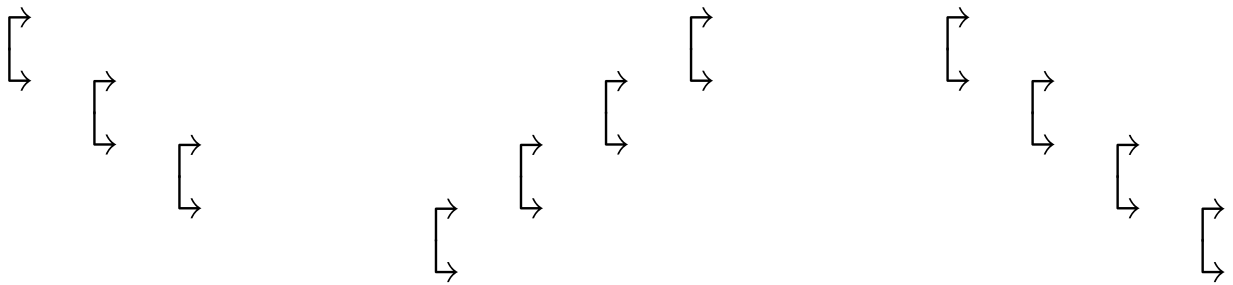


The Bulge Chase



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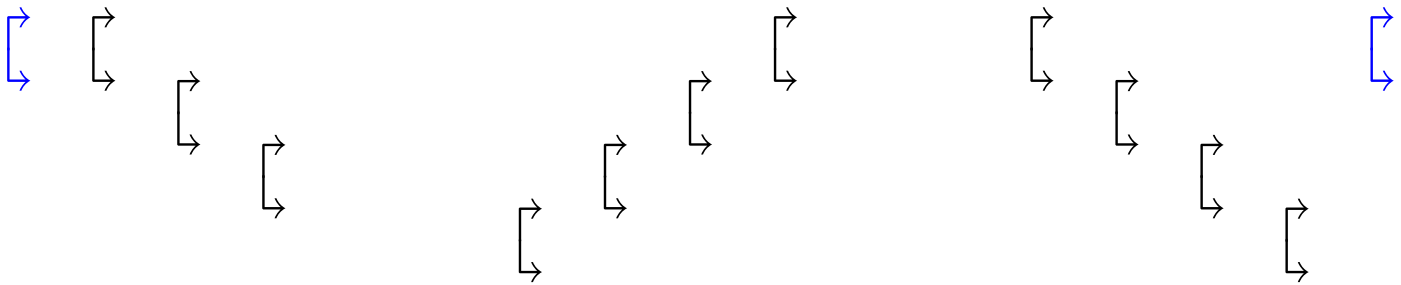

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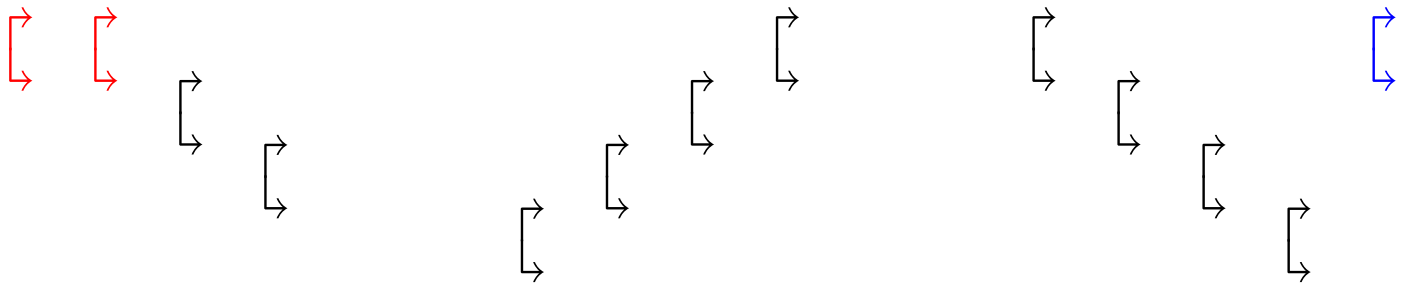

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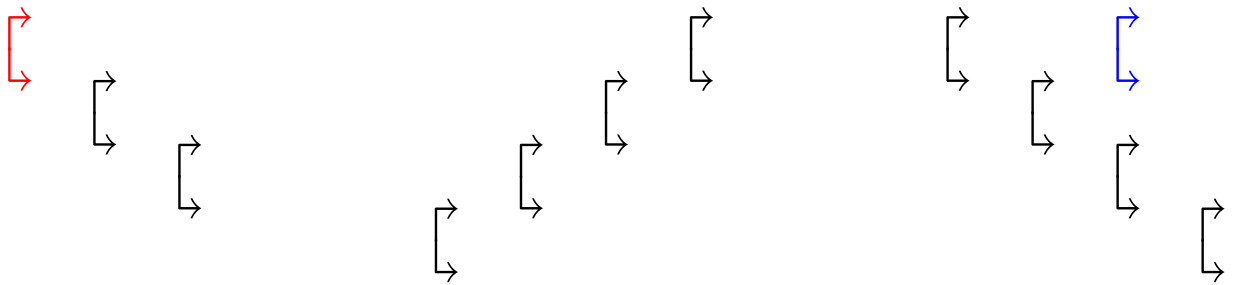

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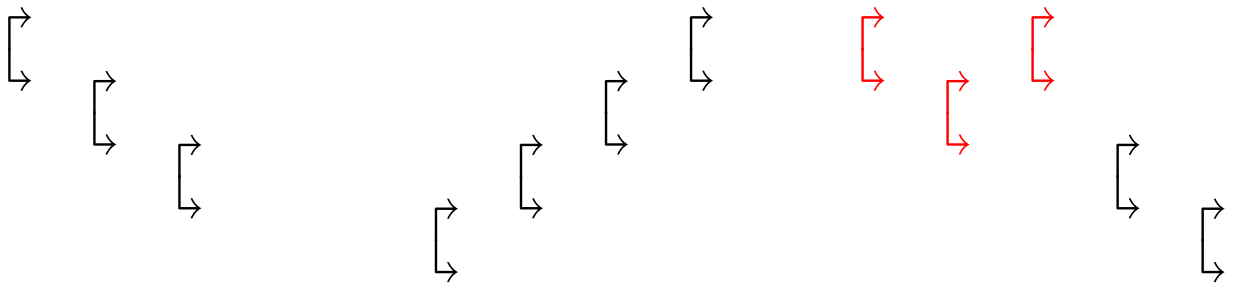

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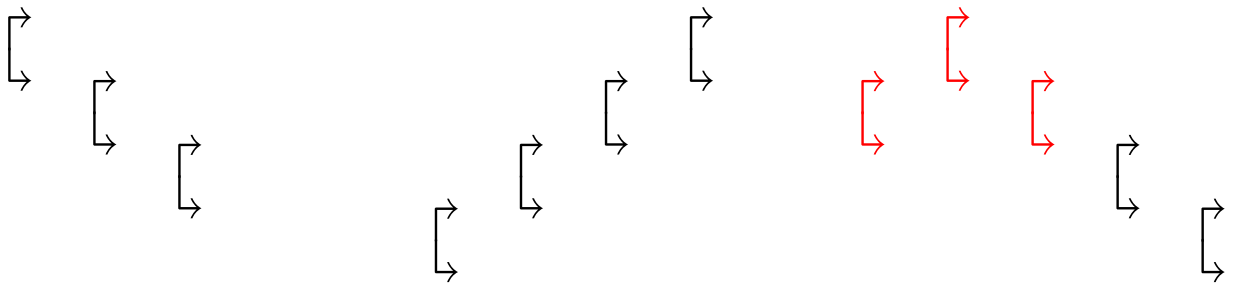

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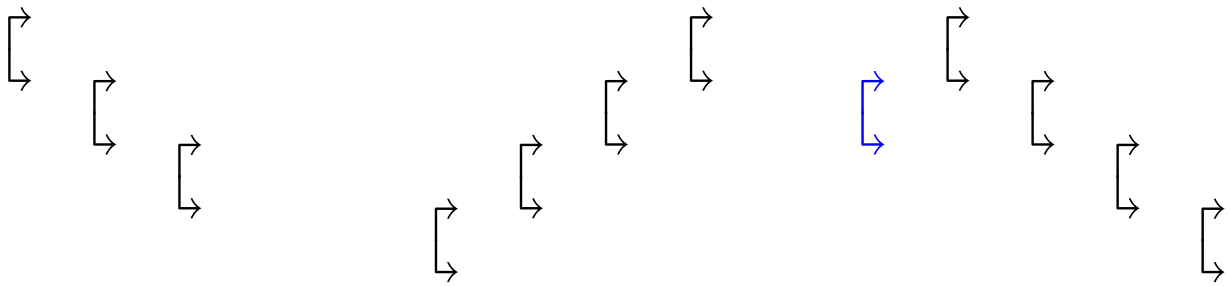

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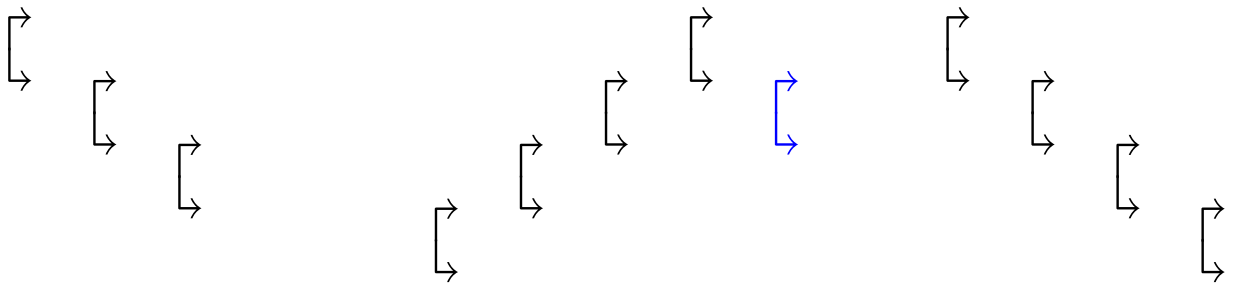

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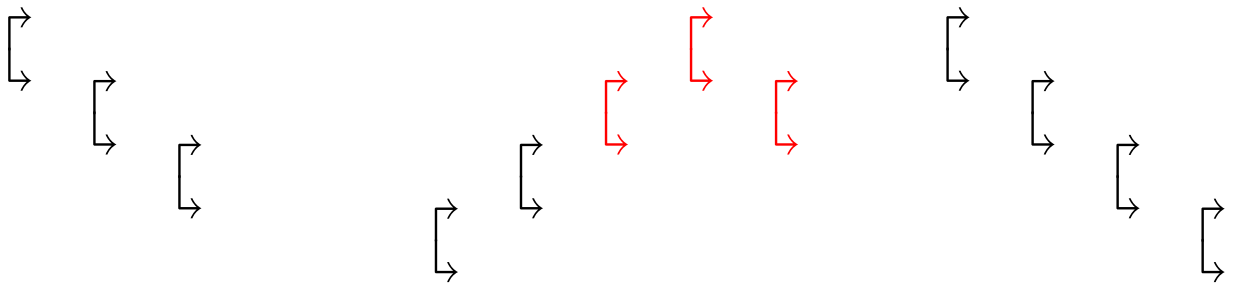

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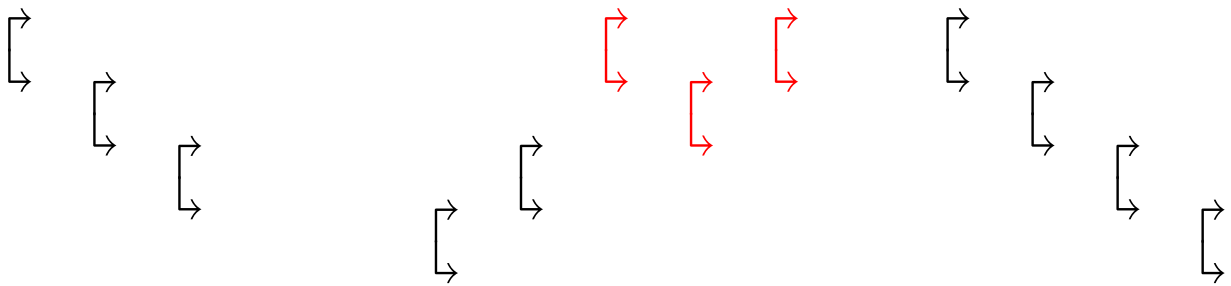

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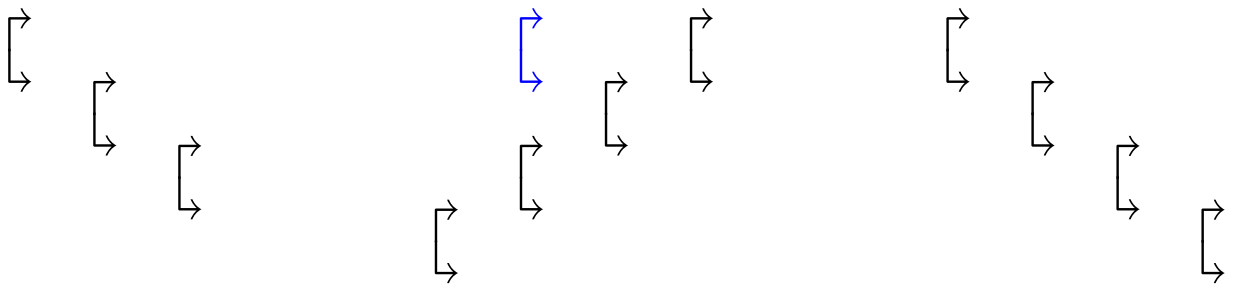

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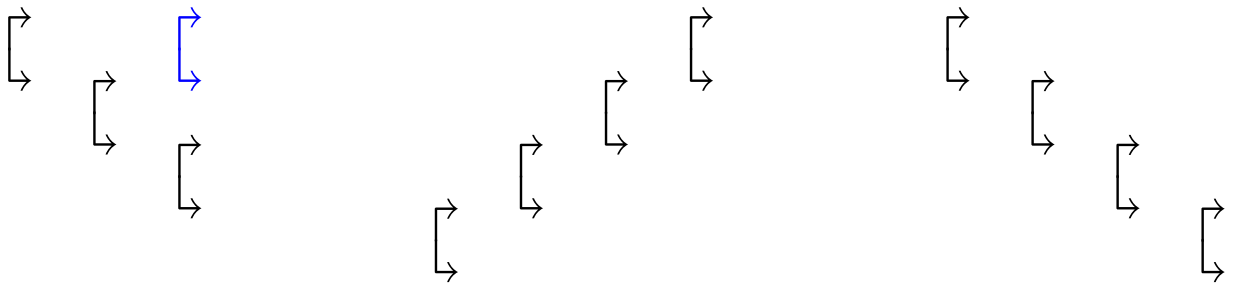

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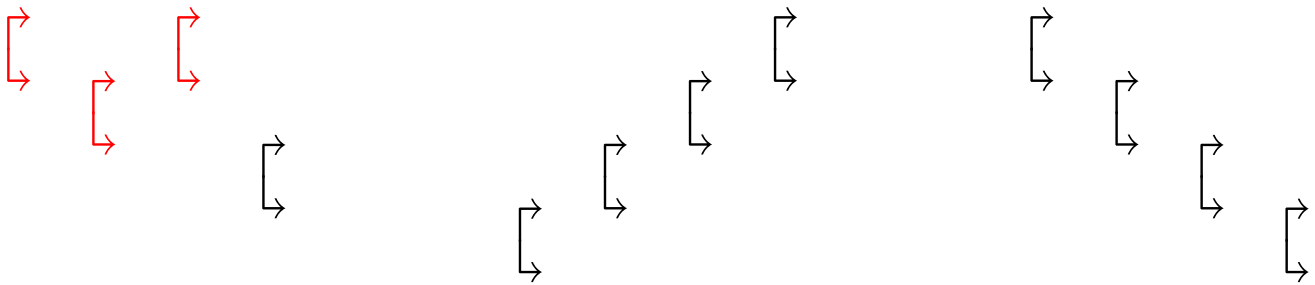

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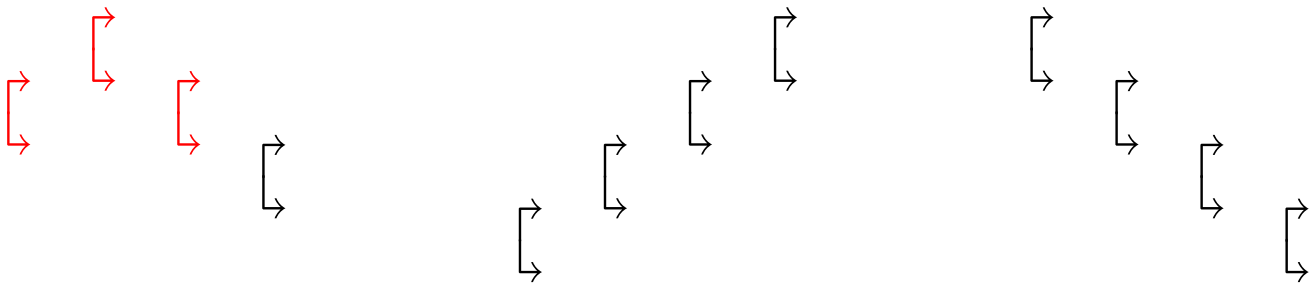

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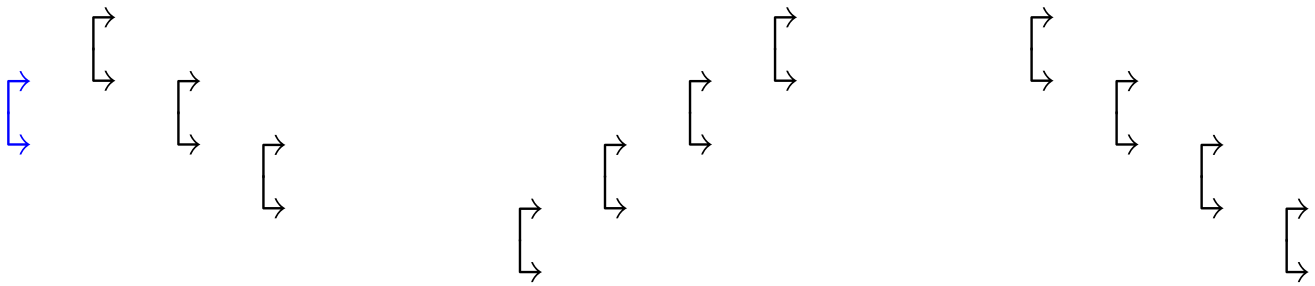

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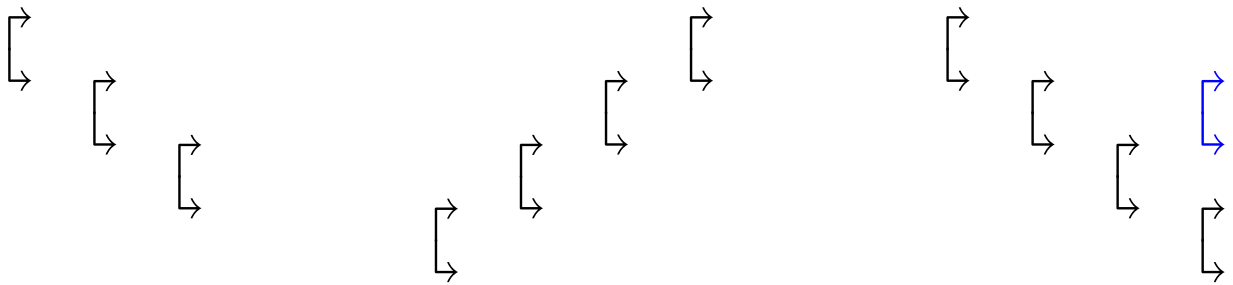

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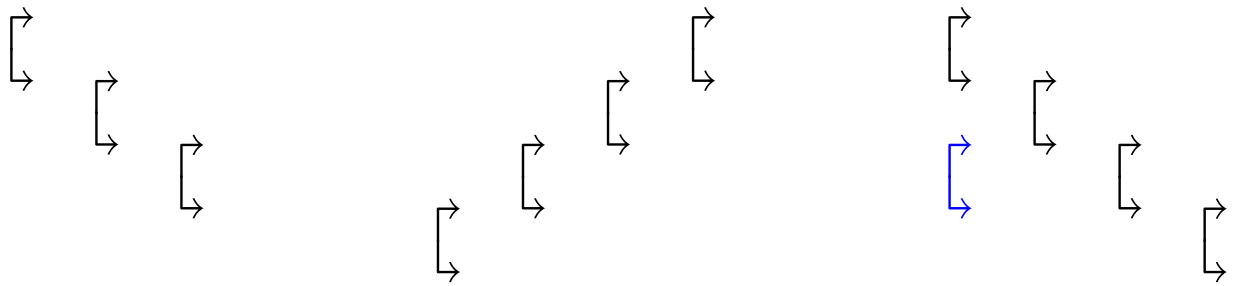

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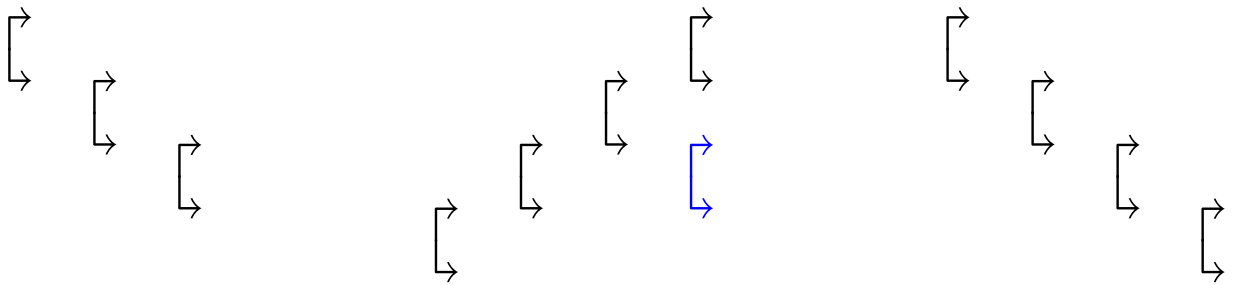

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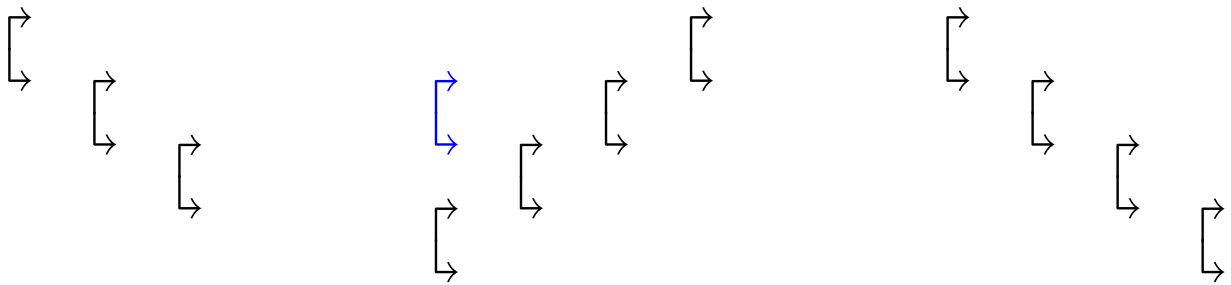

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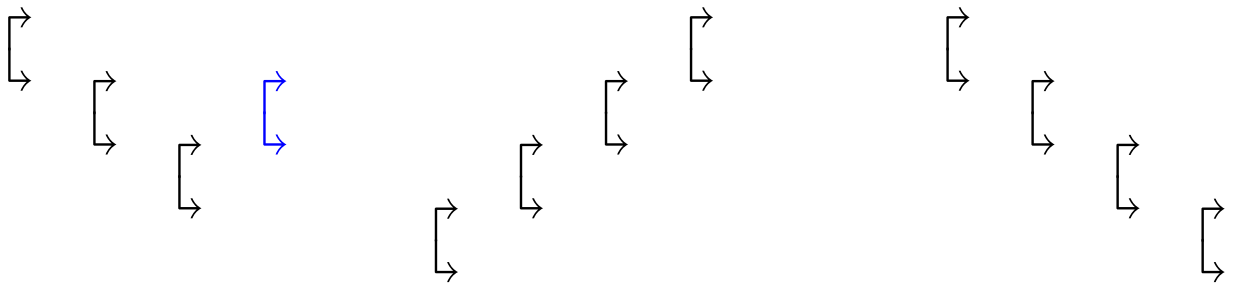

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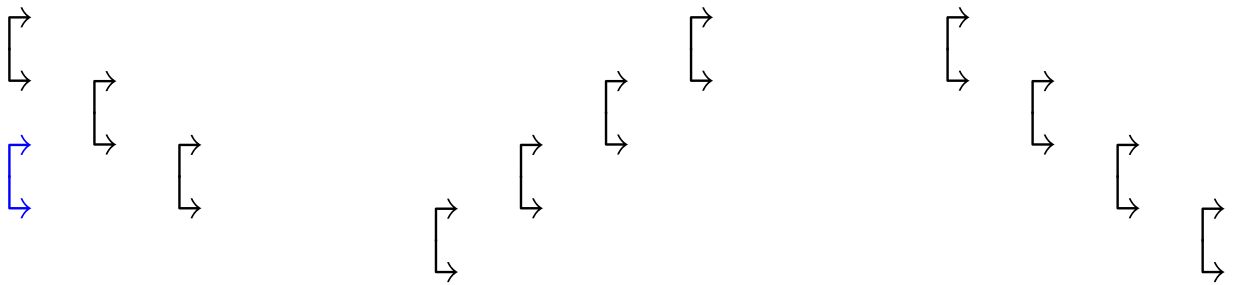

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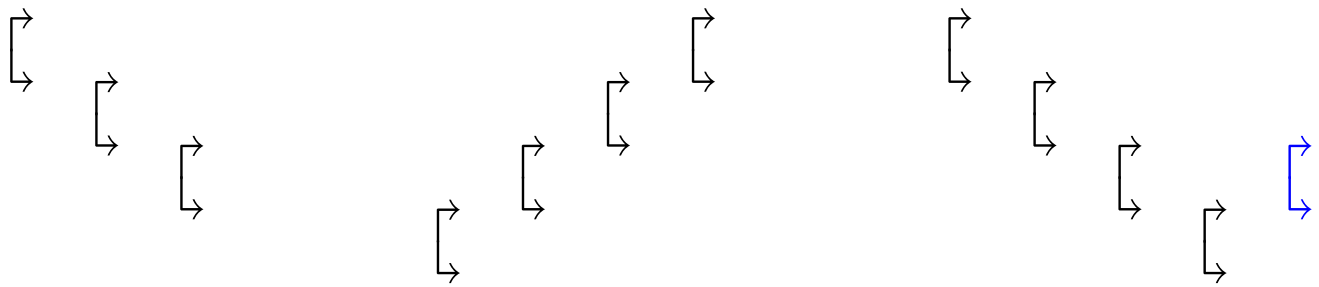

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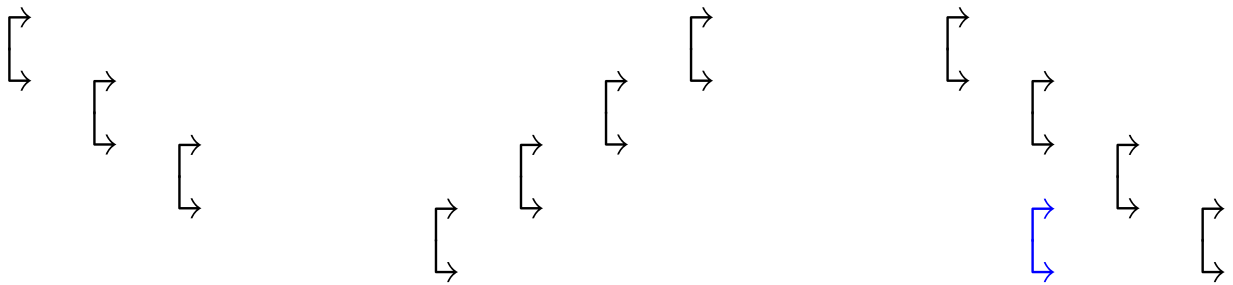

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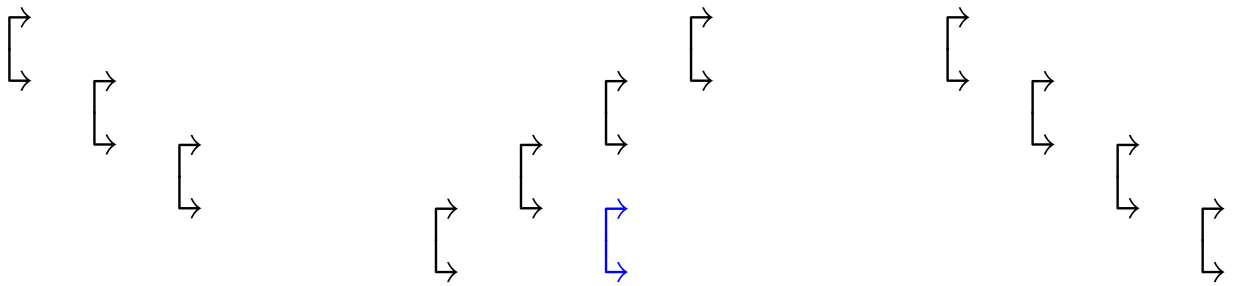

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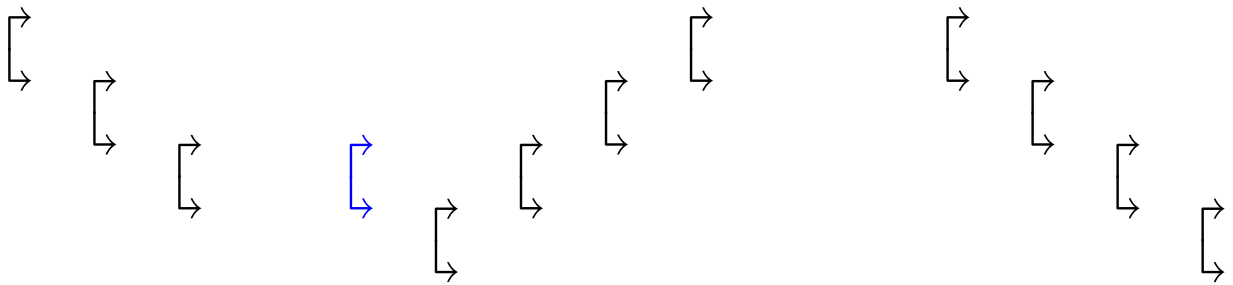

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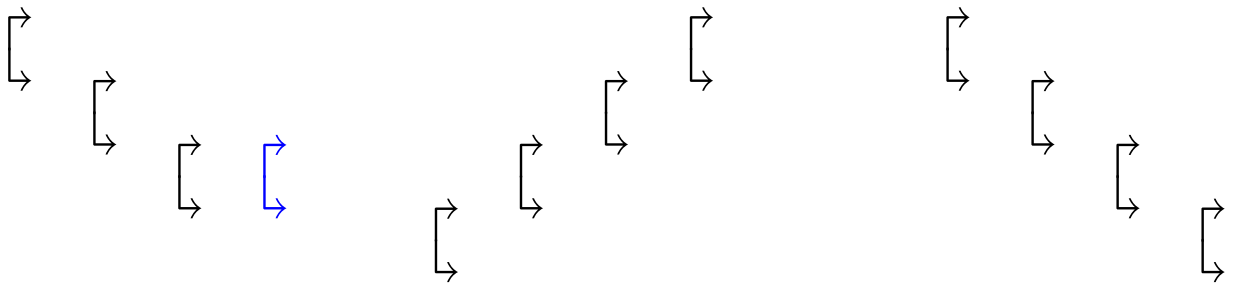

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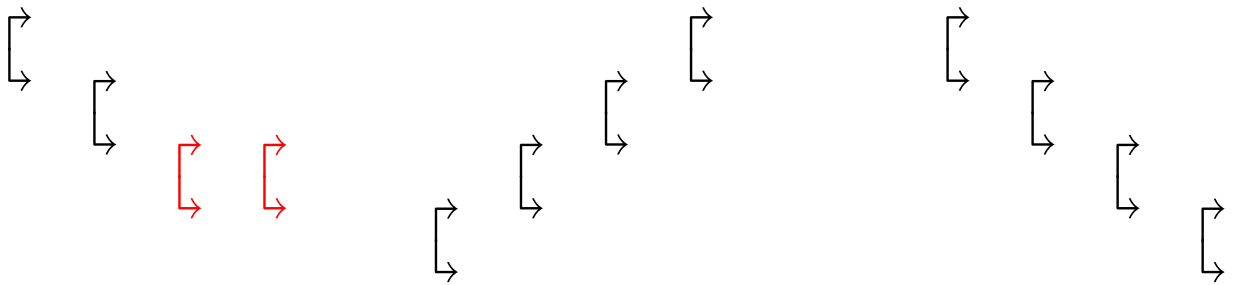

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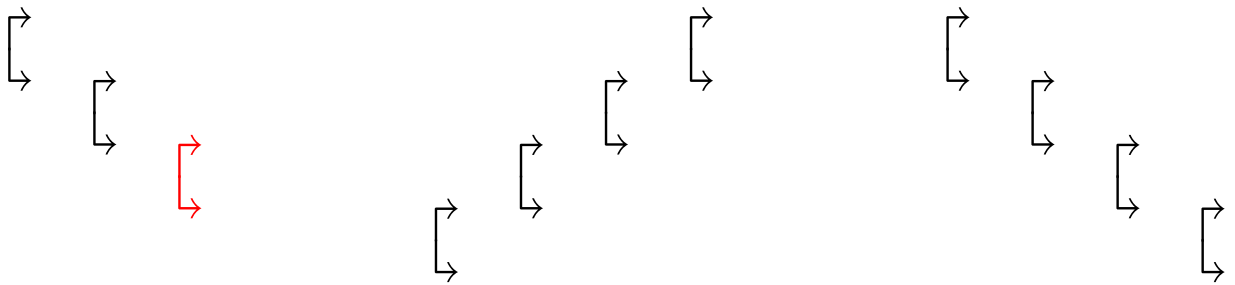

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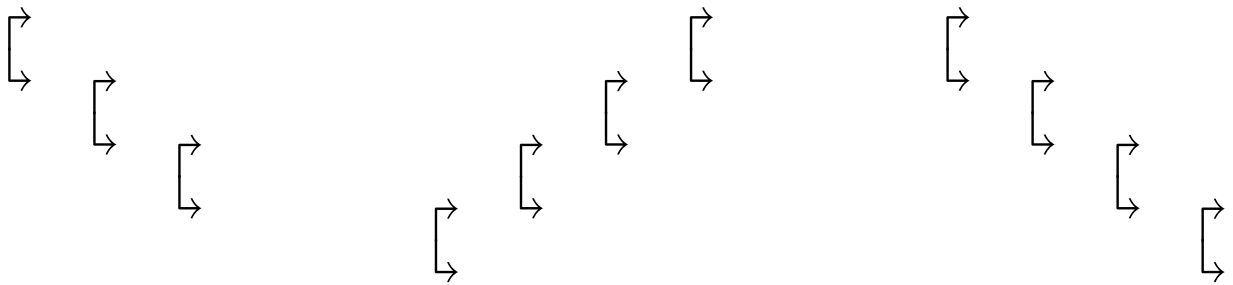

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- iteration complete!
- Cost: $3n$ turnovers/iteration, so $O(n)$ flops/iteration
- Double-shift iteration is similar.
- (Chase two core transformations instead of one.)

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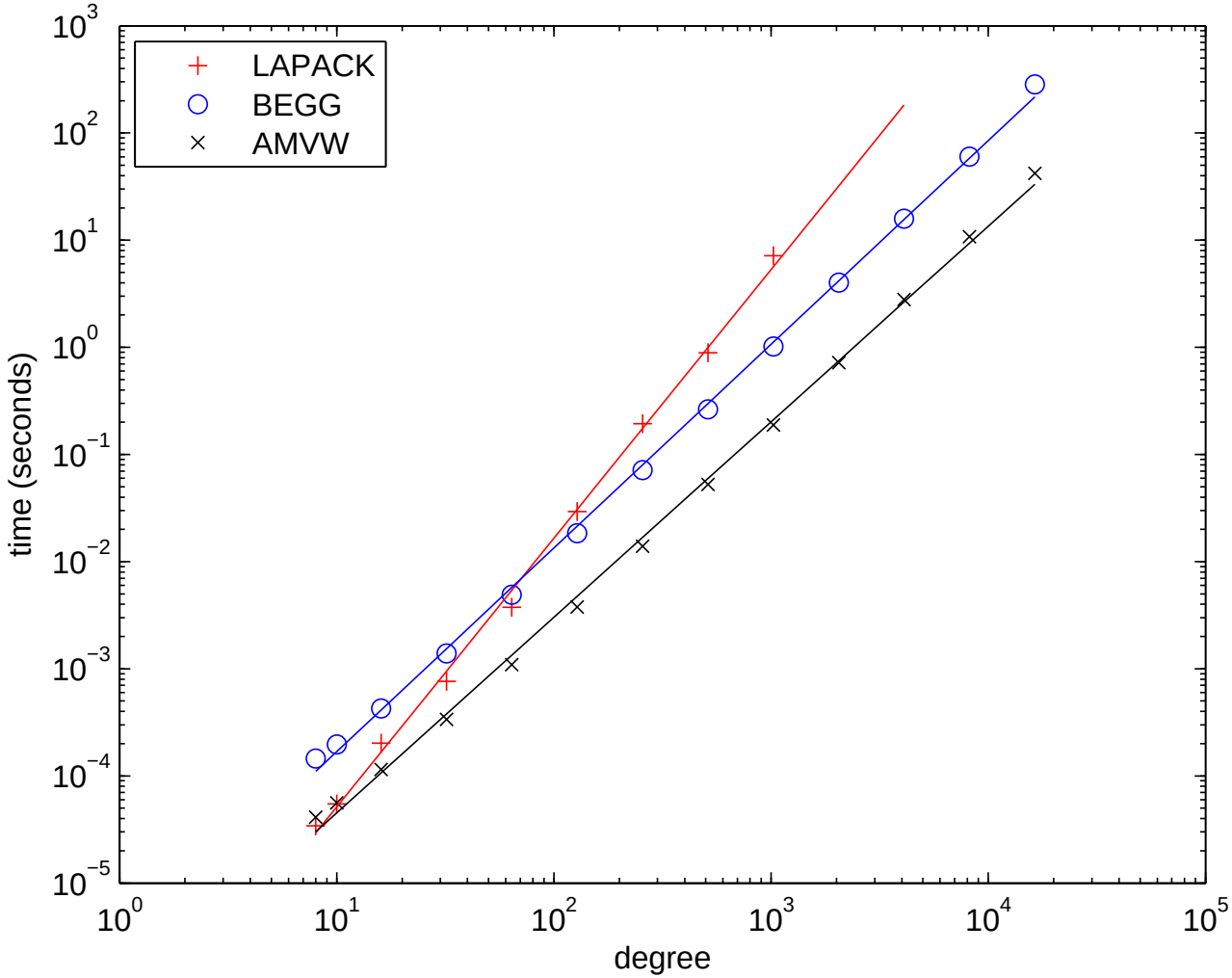
Contestants

- LAPACK code ZHSEQR $(O(n^3))$
- BEGG (Boito et. al. 2012)
- AMVW (our single-shift code)

Speed Comparison, Complex Case



Speed Comparison, Complex Case

- At degree 1024:

method	time
LAPACK	7.14
BEGG	1.02
AMVW	0.18

Speed Comparison, Real Case

- Results similar for double-shift code . . .
- . . . but we're only about twice as fast as the competition.
- (compared with Chandrasekaran et. al.)

Accuracy

- On these big (and well conditioned) problems . . .
- we are about as accurate as LAPACK, . . .
- and more accurate than the other fast methods.
- Our codes act backward stable . . .
- because they are backward stable.
- much more in paper (almost done)
- also tried harder problems (we do OK)

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Thank you for your attention.