Core-Chasing Algorithms for the Eigenvalue Problem

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Department of Mathematics
Washington State University

HHXX, Virginia Tech, June 20, 2017

Our International Research Group

Collaborators:

- **Jared Aurentz**
- **Thomas Mach**
- **Leonardo Robol**
- **Raf Vandebril**

Today's Topic

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  - unitary
  - unitary-plus-rank-one (companion matrix)
  - unitary-plus-low-rank
  - . . . or no special structure

John Francis

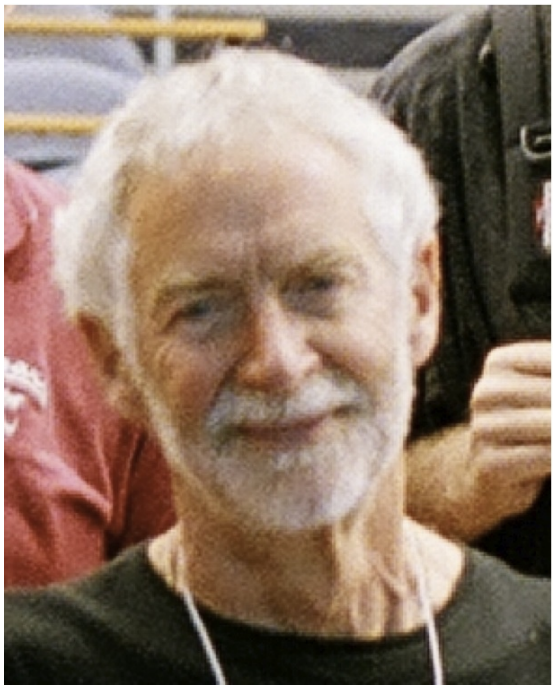

photo: Frank Uhlig, 2009

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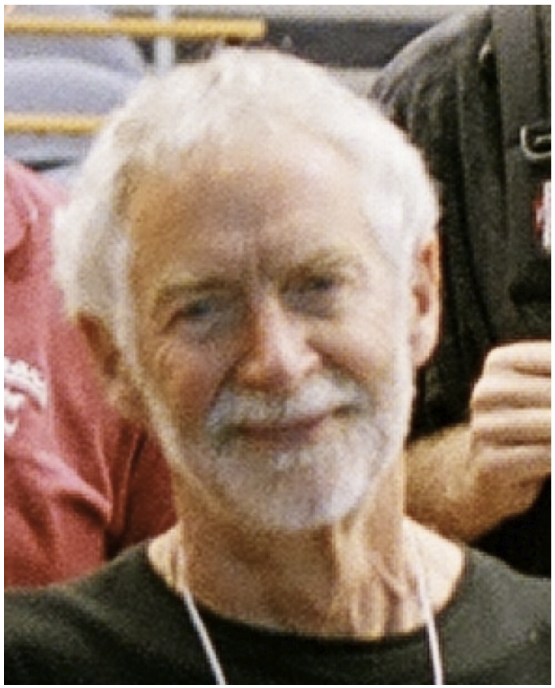

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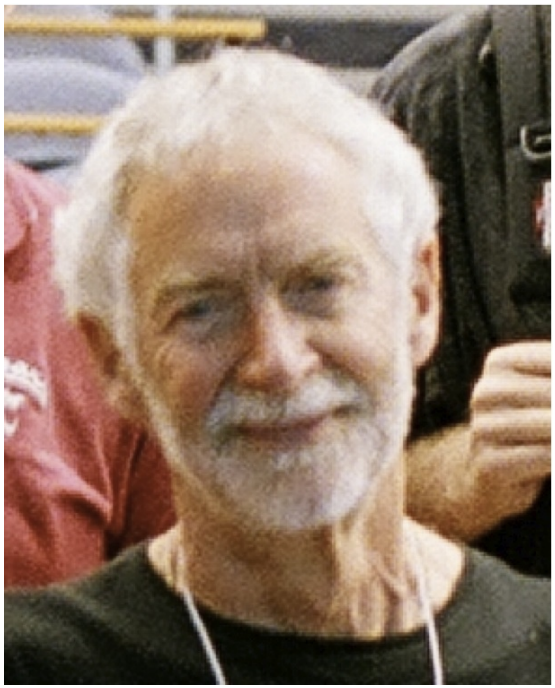

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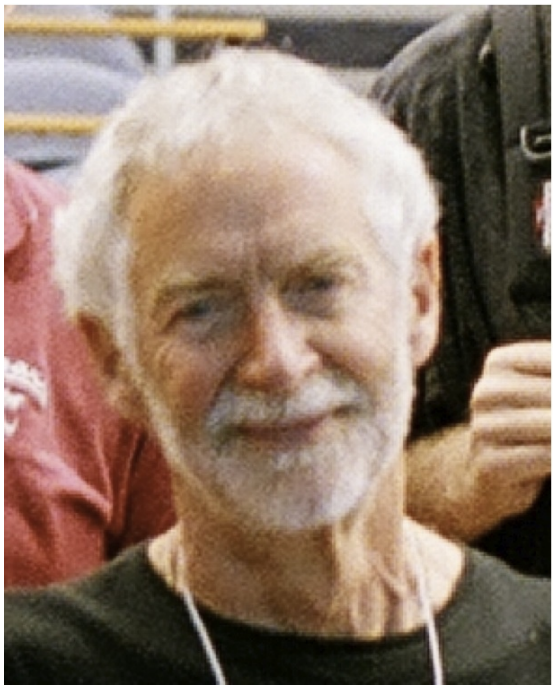

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- invented the winning algorithm in 1959.
- implicitly shifted QR algorithm
- Our algorithms are all variants of this.

- Francis's algorithm . . .

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- We turn it into a core chasing algorithm.

- Francis's algorithm . . .
- . . . is a bulge chasing algorithm.
- We turn it into a core chasing algorithm.
- Instead of chasing bulges, we chase core transformations.

Core Transformations

- What is a core transformation?

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- it's essentially 2×2
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- Ex: Givens rotator, reflector, ...
- We just wanted a generic term.

Core Transformations

$$\begin{bmatrix} 1 & & \\ & \times & \times \\ & \times & \times \end{bmatrix} \begin{bmatrix} \times & \times & \times \\ & \times & \times \\ & \times & \times \end{bmatrix} = \begin{bmatrix} \times & \times & \times \\ & \times & \times \\ & 0 & \times \end{bmatrix}$$

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Abbreviated notation

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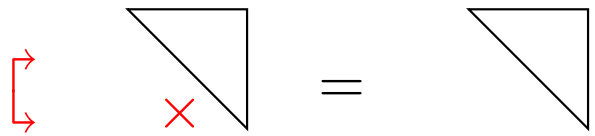
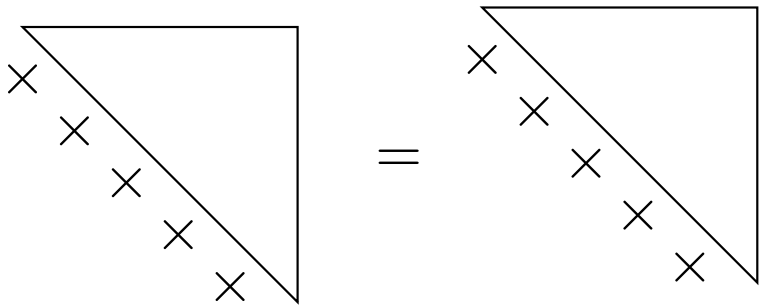


Hessenberg QR decomposition



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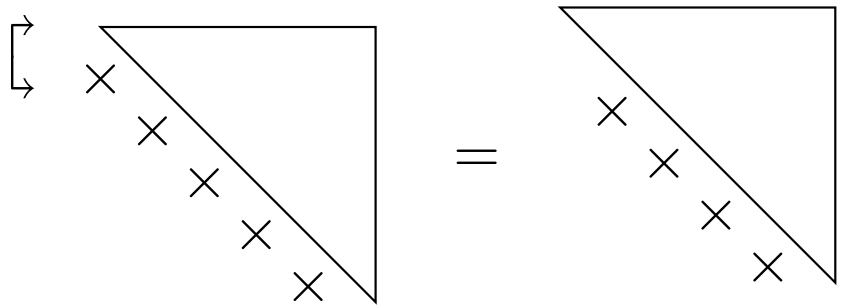

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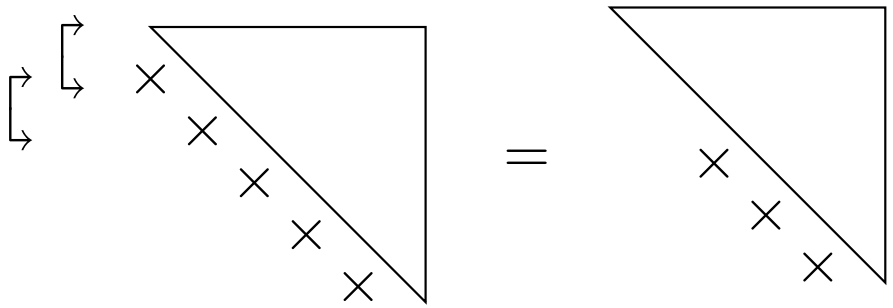

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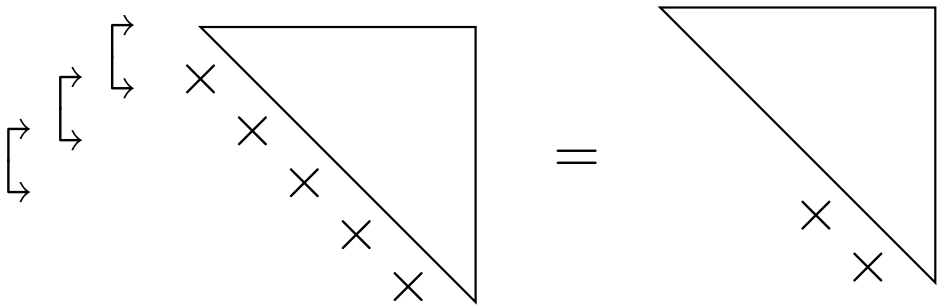

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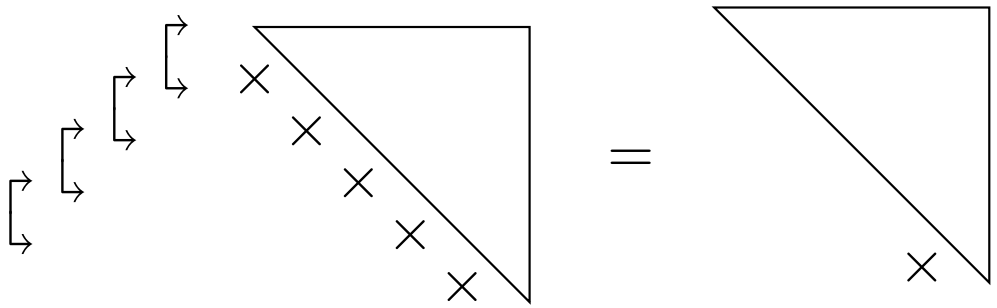

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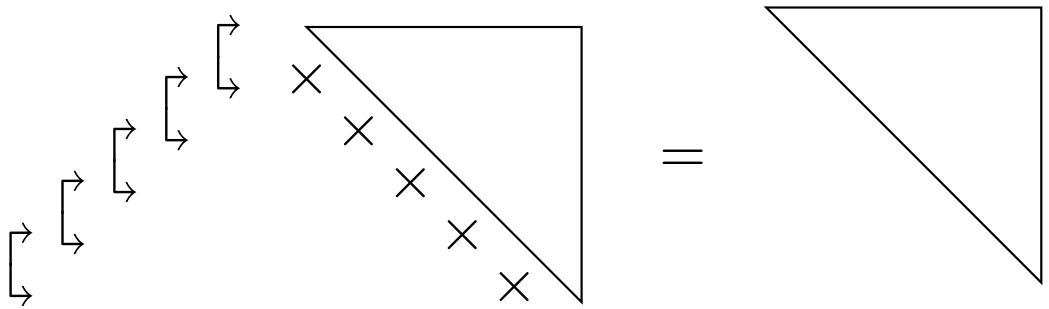

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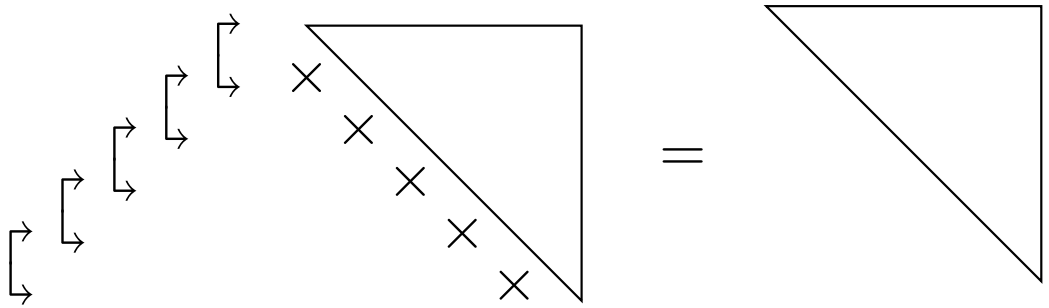
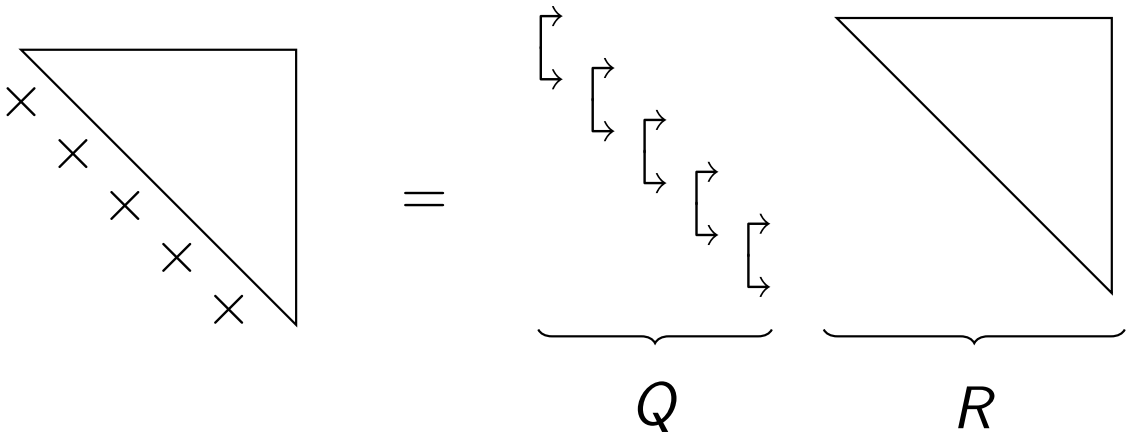


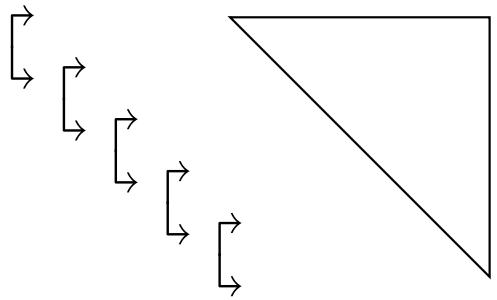
Now invert the core transformations.

Hessenberg QR decomposition



Our algorithms operate on the matrix in QR decomposed form.

$$A = QR =$$

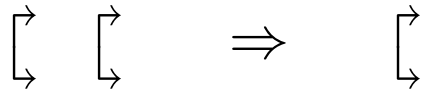


This is not inefficient.

We apply Francis's algorithm to this factored form.

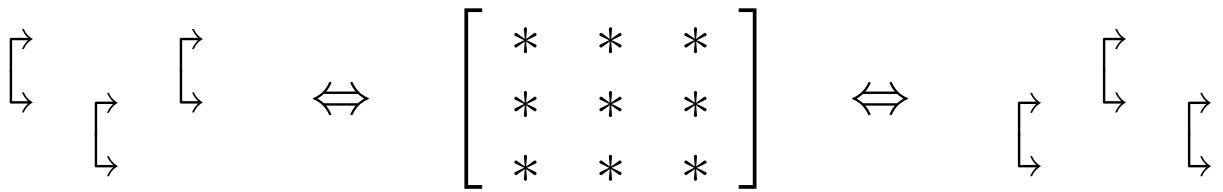
Operating on Core Transformations

Fusion



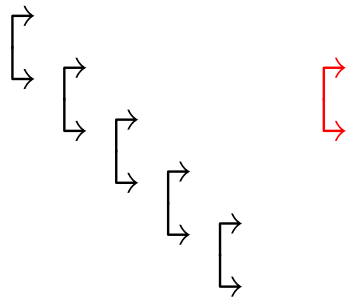
Operating on Core Transformations

Turnover



Operating on Core Transformations

Turnover as a shift-through operation



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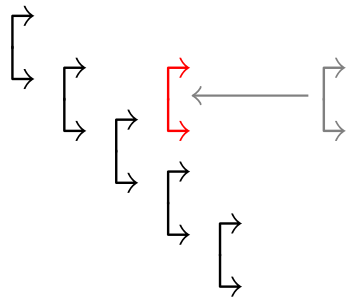

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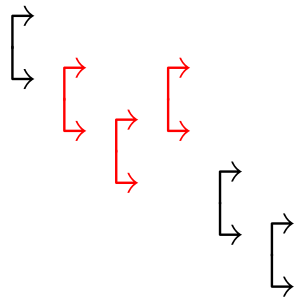

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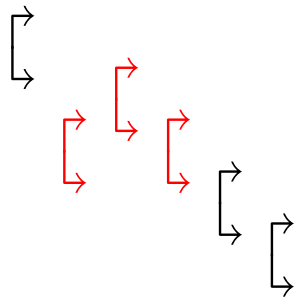

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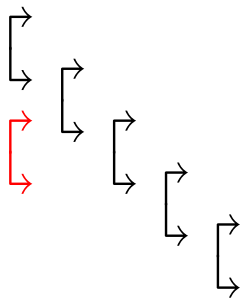

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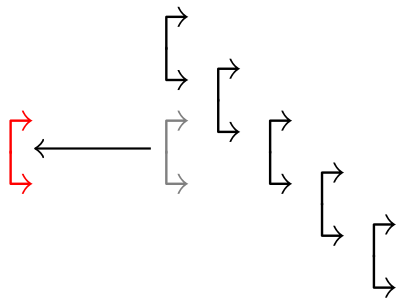

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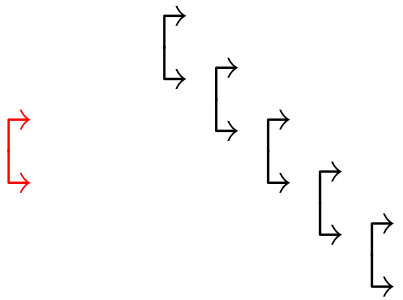
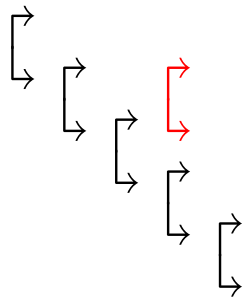


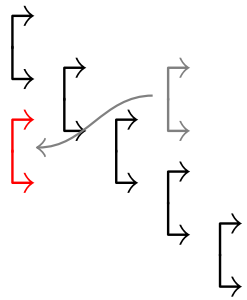
Operating on Core Transformations

Turnover as a shift-through operation : Abbreviated notation



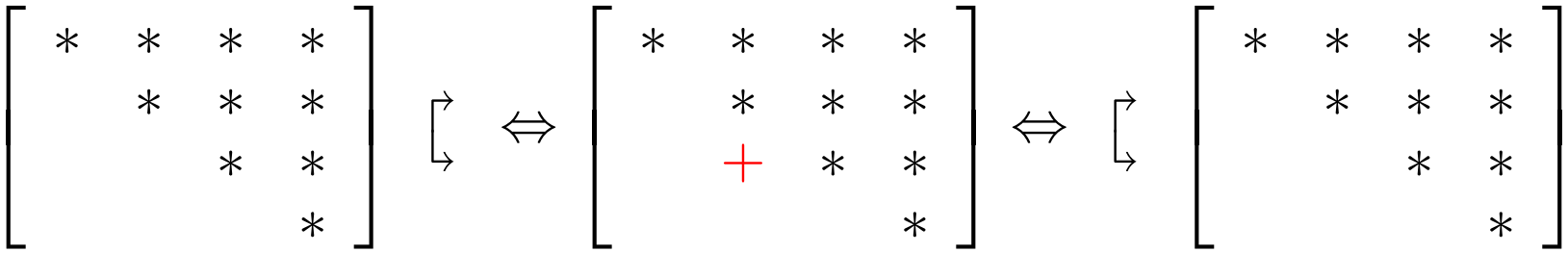
Operating on Core Transformations

Turnover as a shift-through operation : Abbreviated notation



Operating on Core Transformations

Passing a core transformation through a triangular matrix

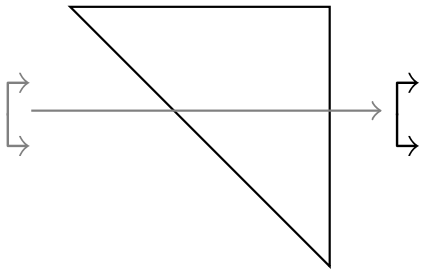


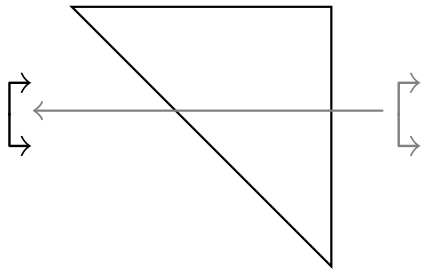
Cost is $O(n)$ flops.

Operating on Core Transformations

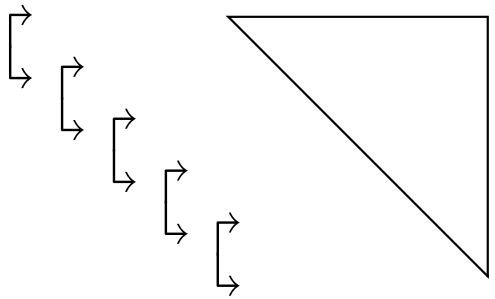
Passing a core transformation through a triangular matrix

Abbreviated Notation:





Core Chasing



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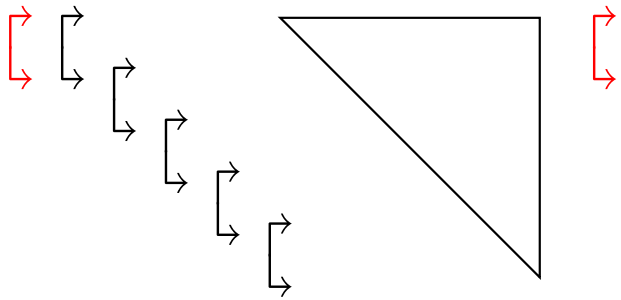

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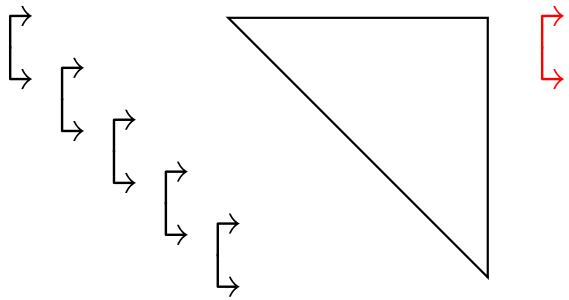

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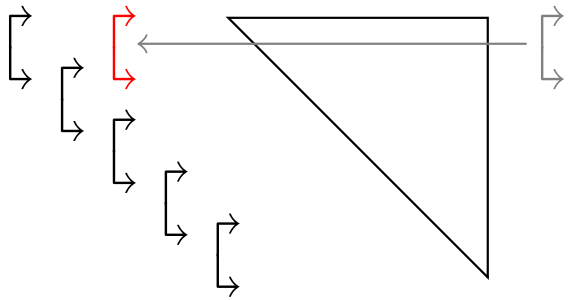

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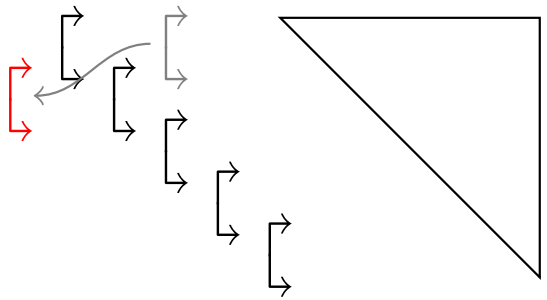

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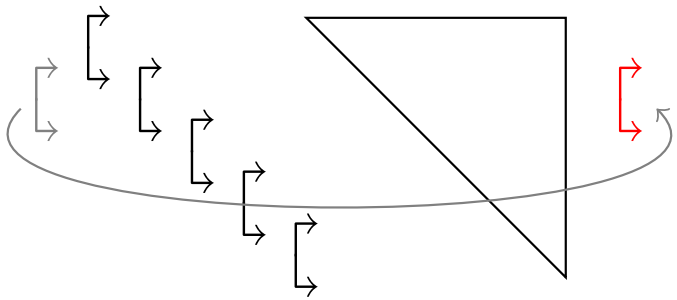

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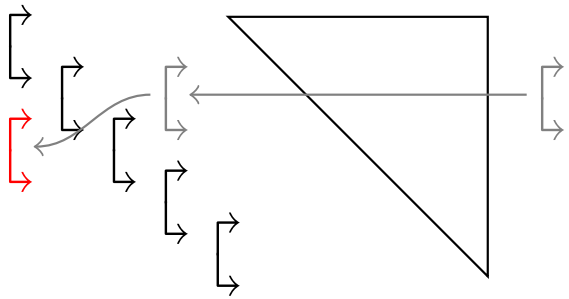

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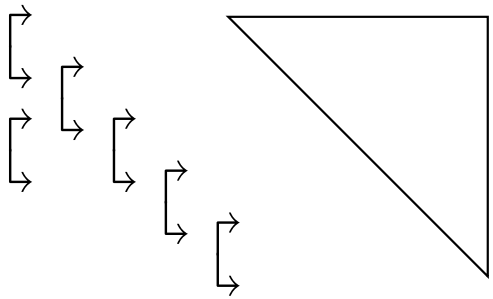

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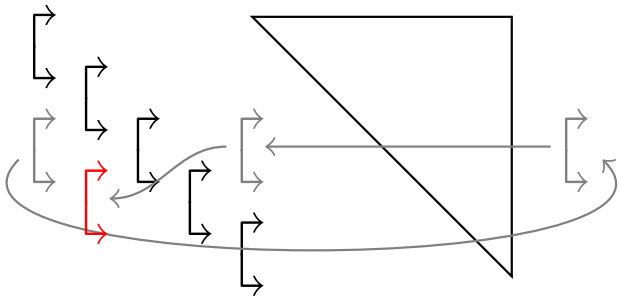

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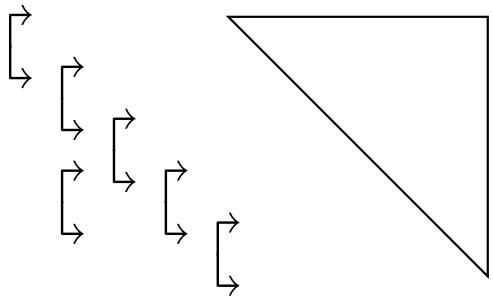

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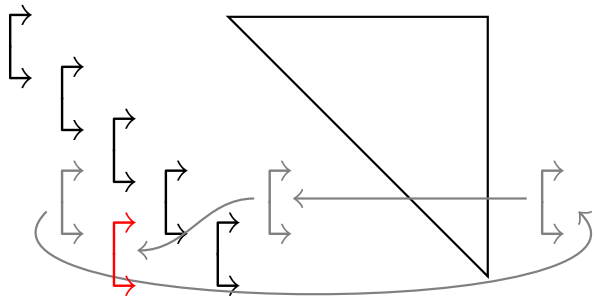

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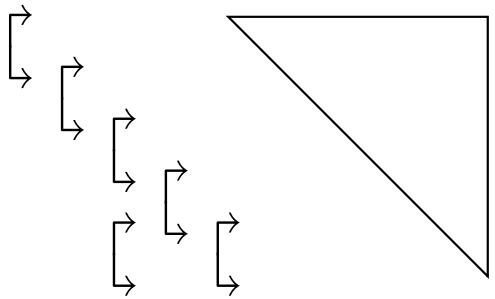

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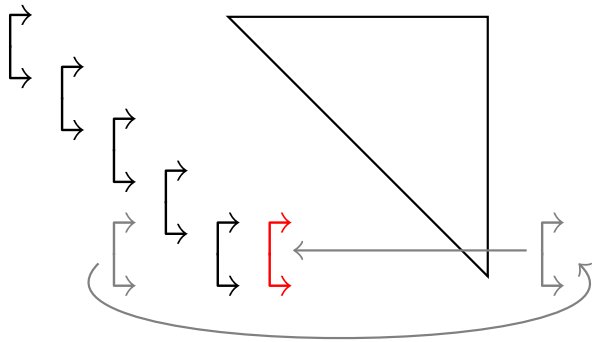

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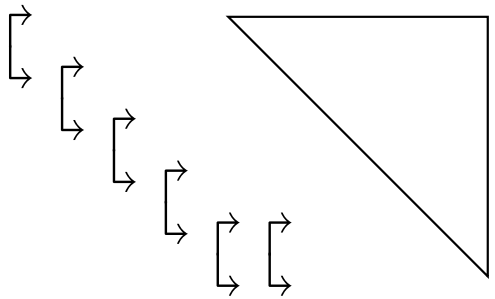

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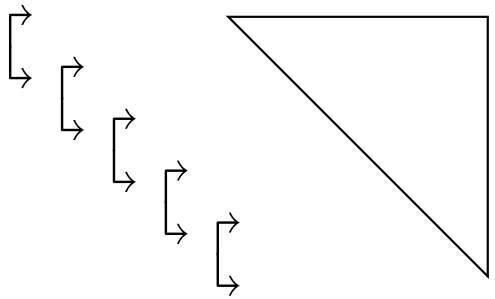


Cost

Flop Count

Cost

- $O(n^3)$ total flops
- $O(n^2)$ storage
- about the same as for standard Francis iteration.

Advantages

Are there any advantages?

Advantages

Are there any advantages?

- superior deflation procedure

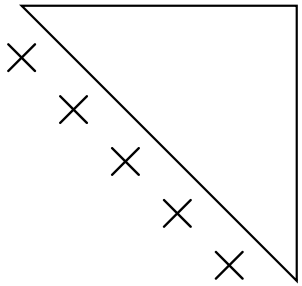
Advantages

Are there any advantages?

- superior deflation procedure
- some structured cases

Deflation

Standard deflation criterion:



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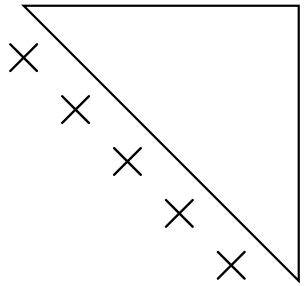


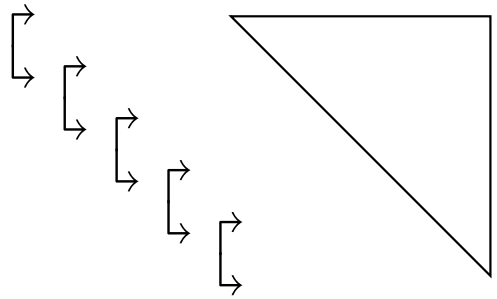
Set $a_{j+1,j}$ to zero if

$$|a_{j+1,j}| < u\,(|a_{j,j}| + |a_{j+1,j+1}|).$$

(u is *unit roundoff.*)

Deflation

Our deflation criterion:



$$Q_j = \begin{bmatrix} I & & & \\ & c_j & -s_j & \\ & s_j & \overline{c}_j & \\ & & & I \end{bmatrix}$$

Deflation

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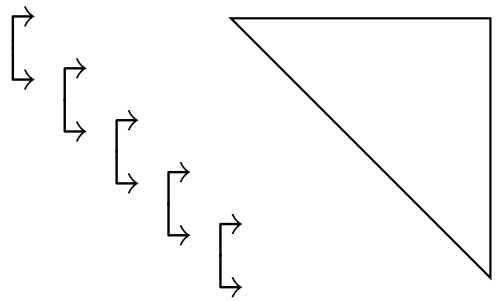

$$Q_j = \begin{bmatrix} I & & & \\ & c_j & -s_j & \\ & s_j & \overline{c}_j & \\ & & & I \end{bmatrix}$$

Set s_j to zero if $\quad |s_j| < u.$

(u is *unit roundoff.*)

Deflation

- Both criteria are normwise backward stable.

Deflation

- Both criteria are normwise backward stable.
- How does this affect eigenvalues?

Deflation

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Structures we can exploit

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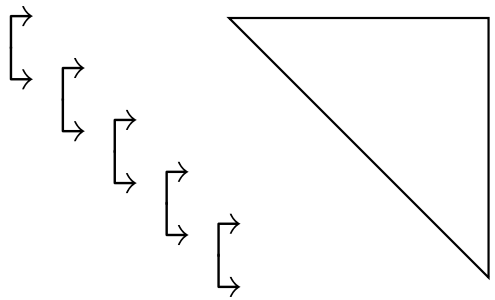
$A = QR =$ 

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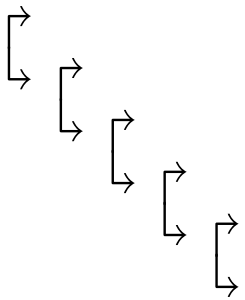
$A = QR =$ 

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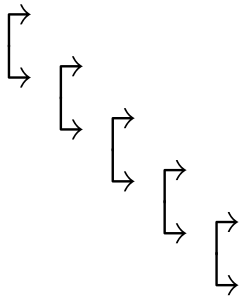
$A = QR =$ 

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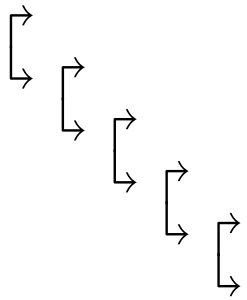
$A = QR =$ 

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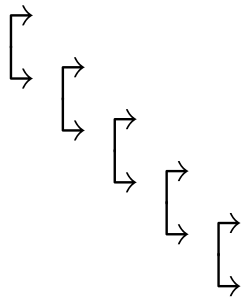
$A = QR =$ 

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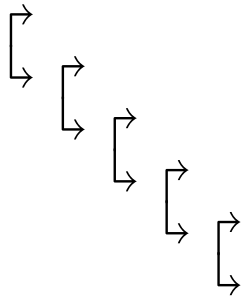
$A = QR =$ 

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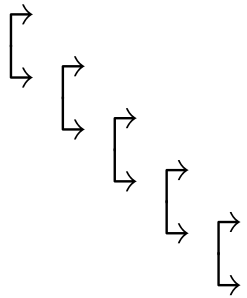
$A = QR =$ 

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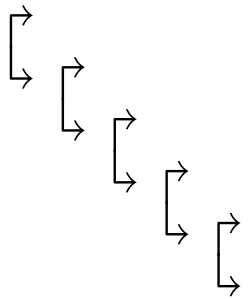
$A = QR =$ 

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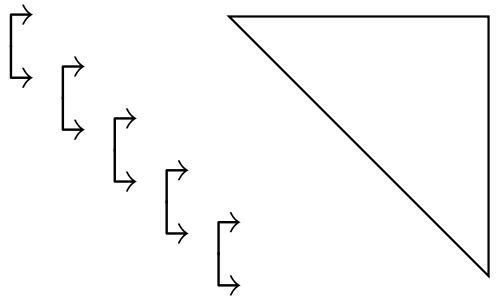
  - I spoke about this at the previous Householder symposium.

Representation of R

We store the QR decomposed form.

$A = QR =$ 

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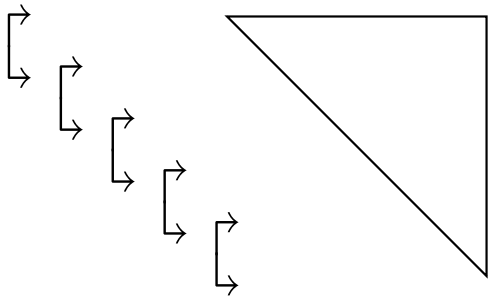
$$A = QR = \quad$$ 

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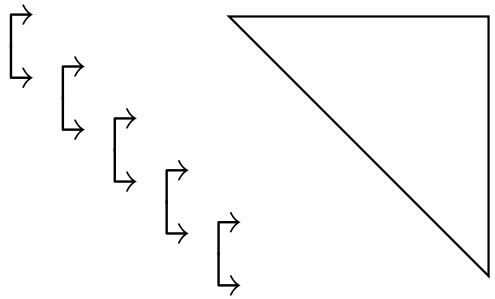
$$A = QR = \quad$$ 

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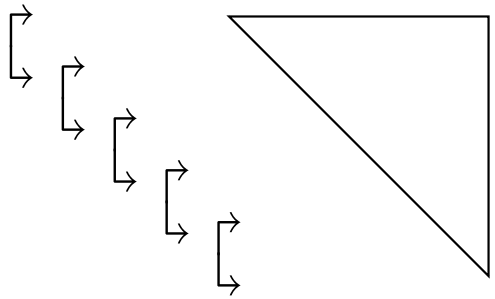
$$A = QR = \quad$$ 

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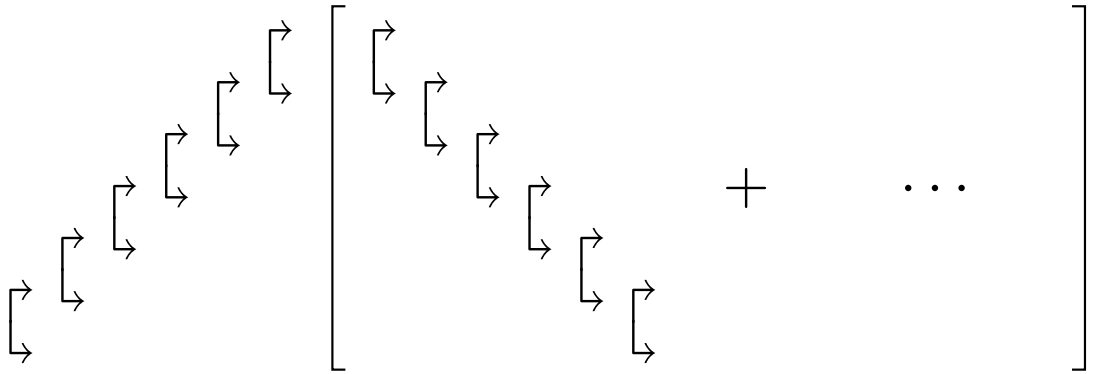

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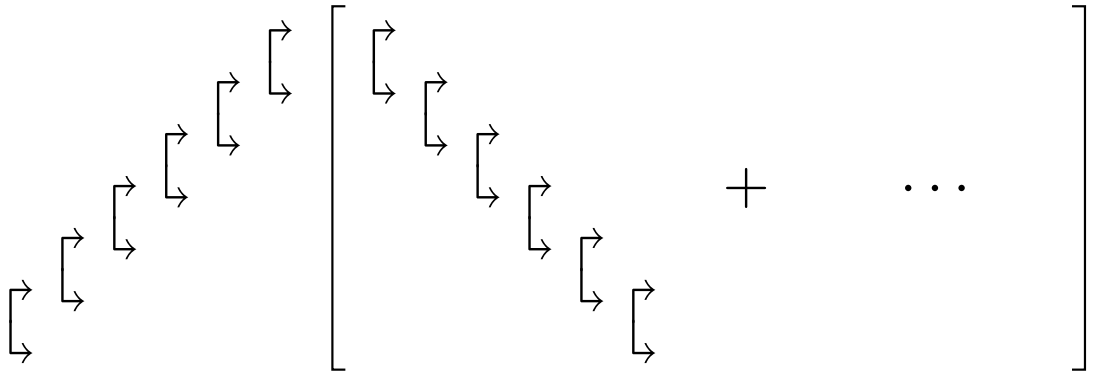

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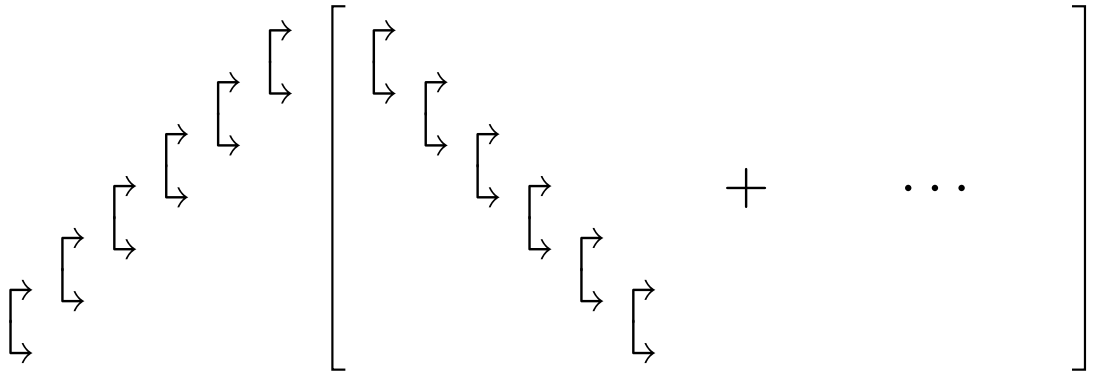

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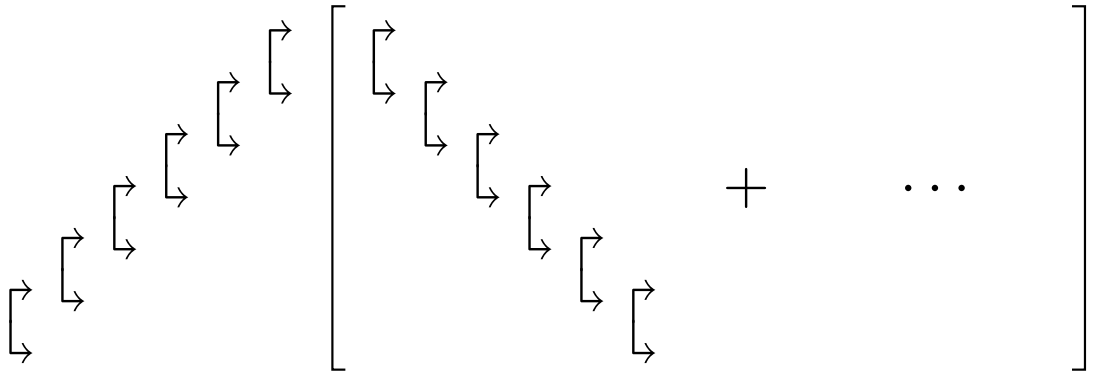

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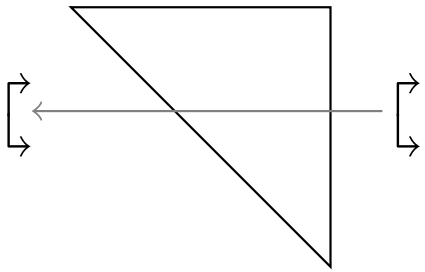
Passing a core transformation through R



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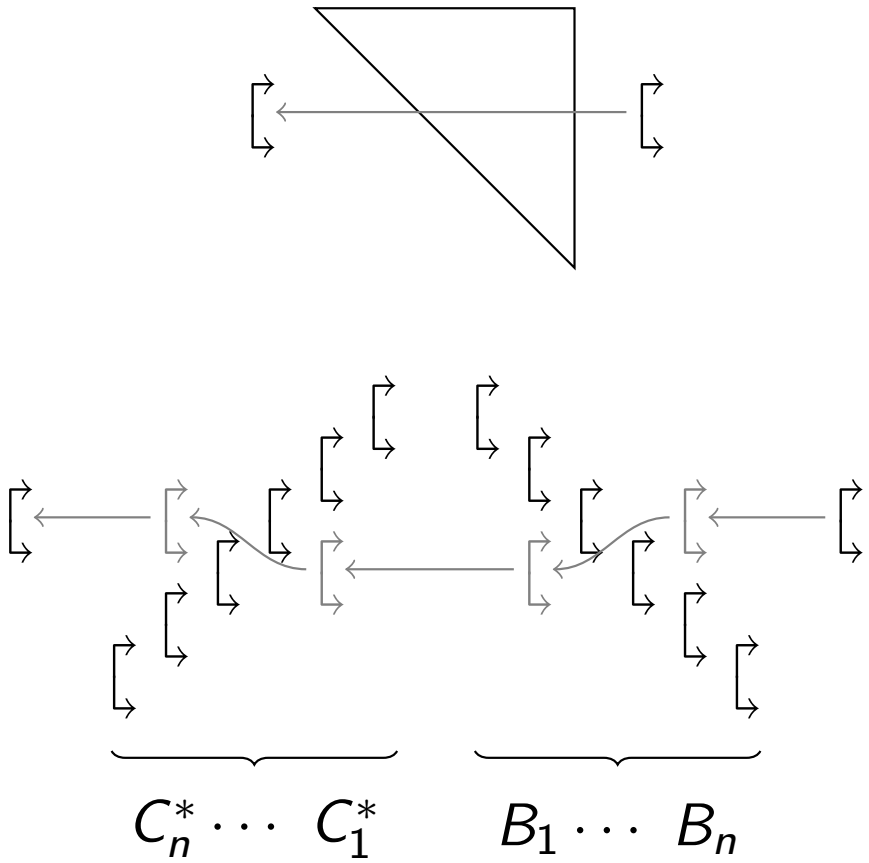

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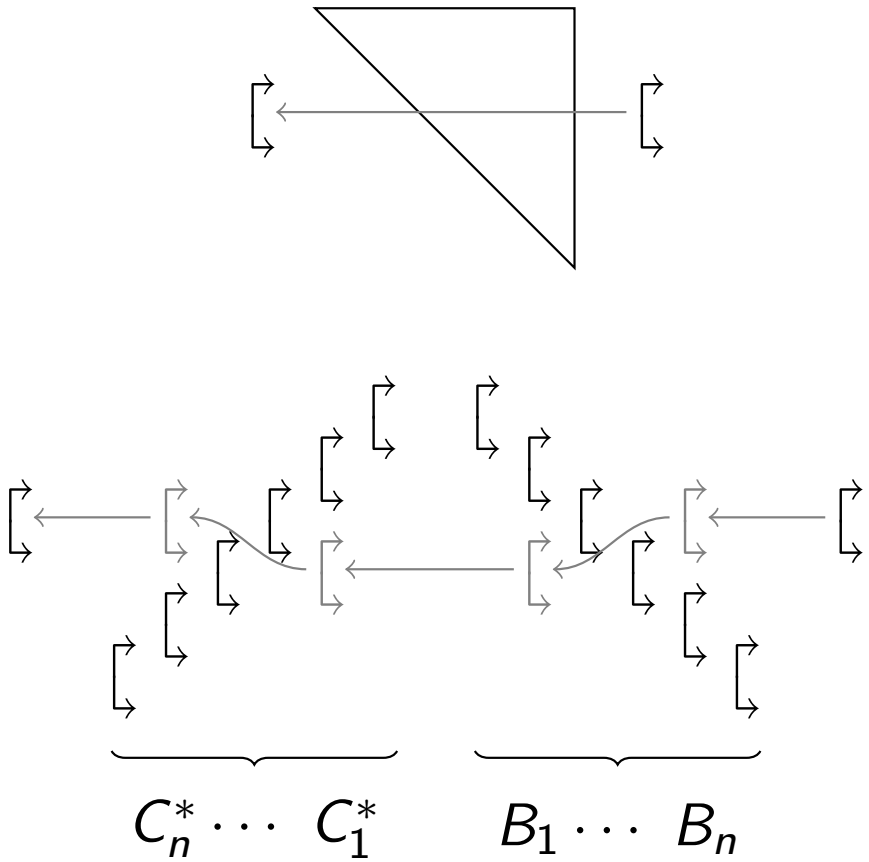

Cost: $O(1)$ flops instead of $O(n)$.

Other things we can do

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Monograph in progress (130+ pp.)

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- e.g. if a_n is tiny.

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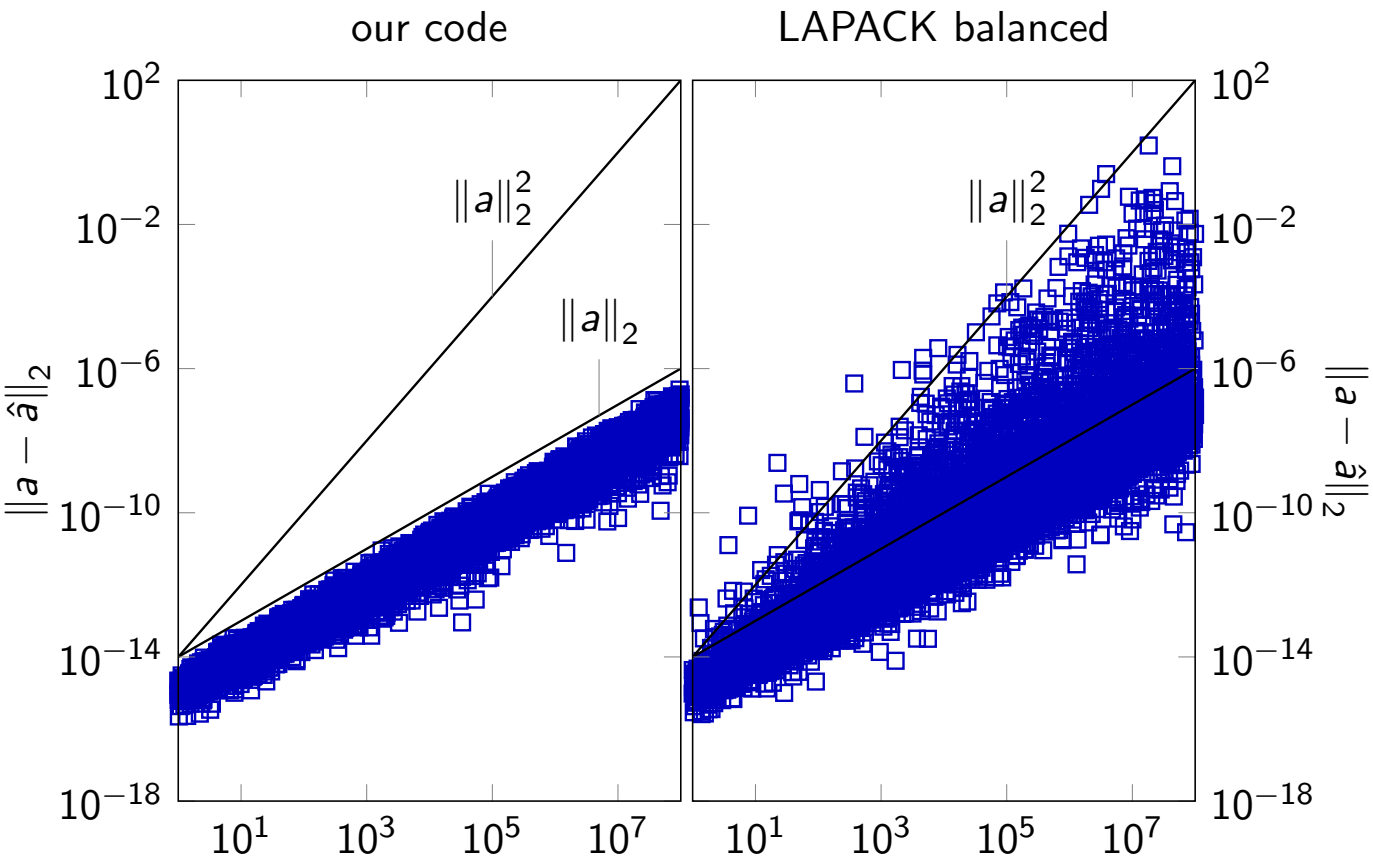
Nice Picture



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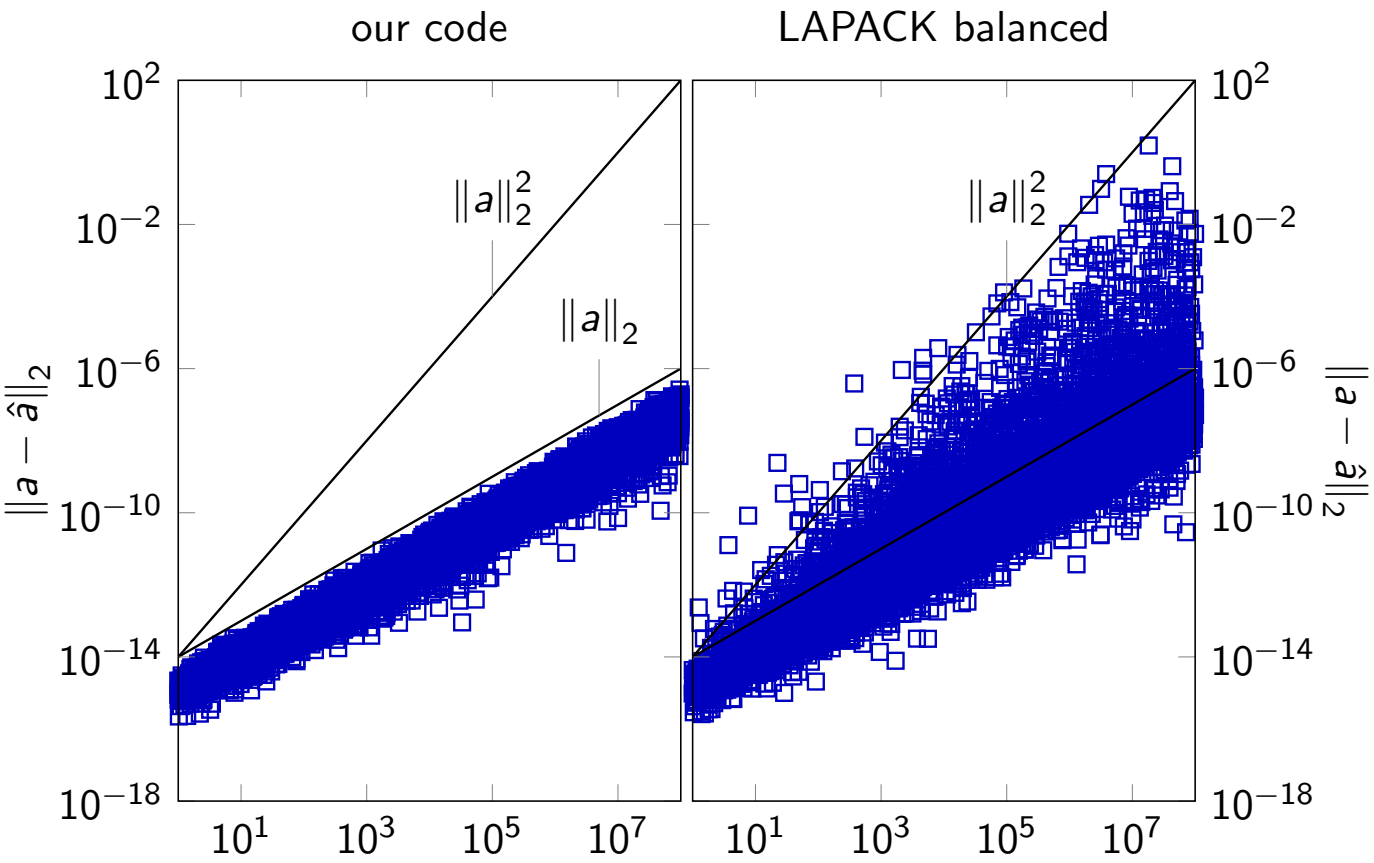

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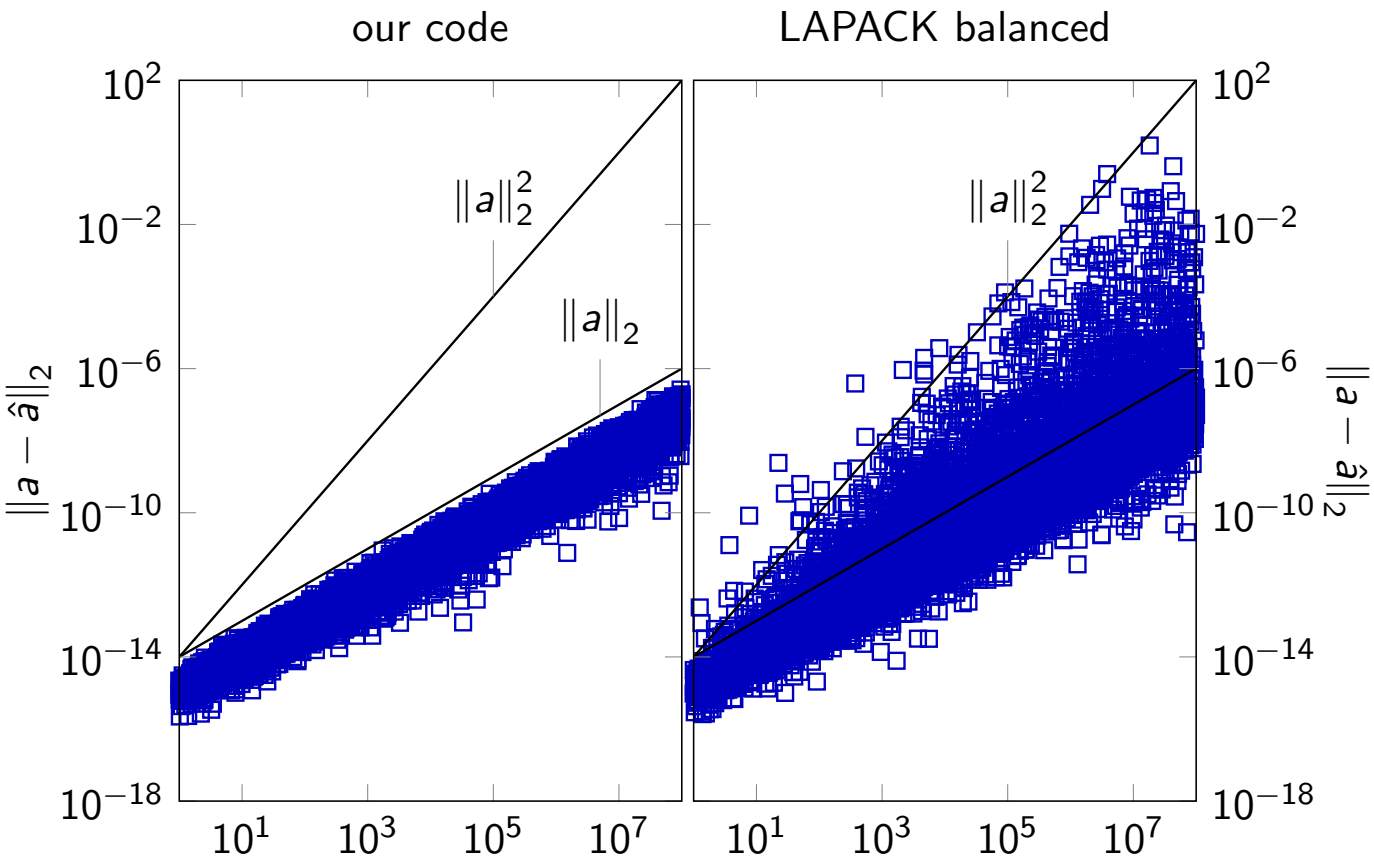

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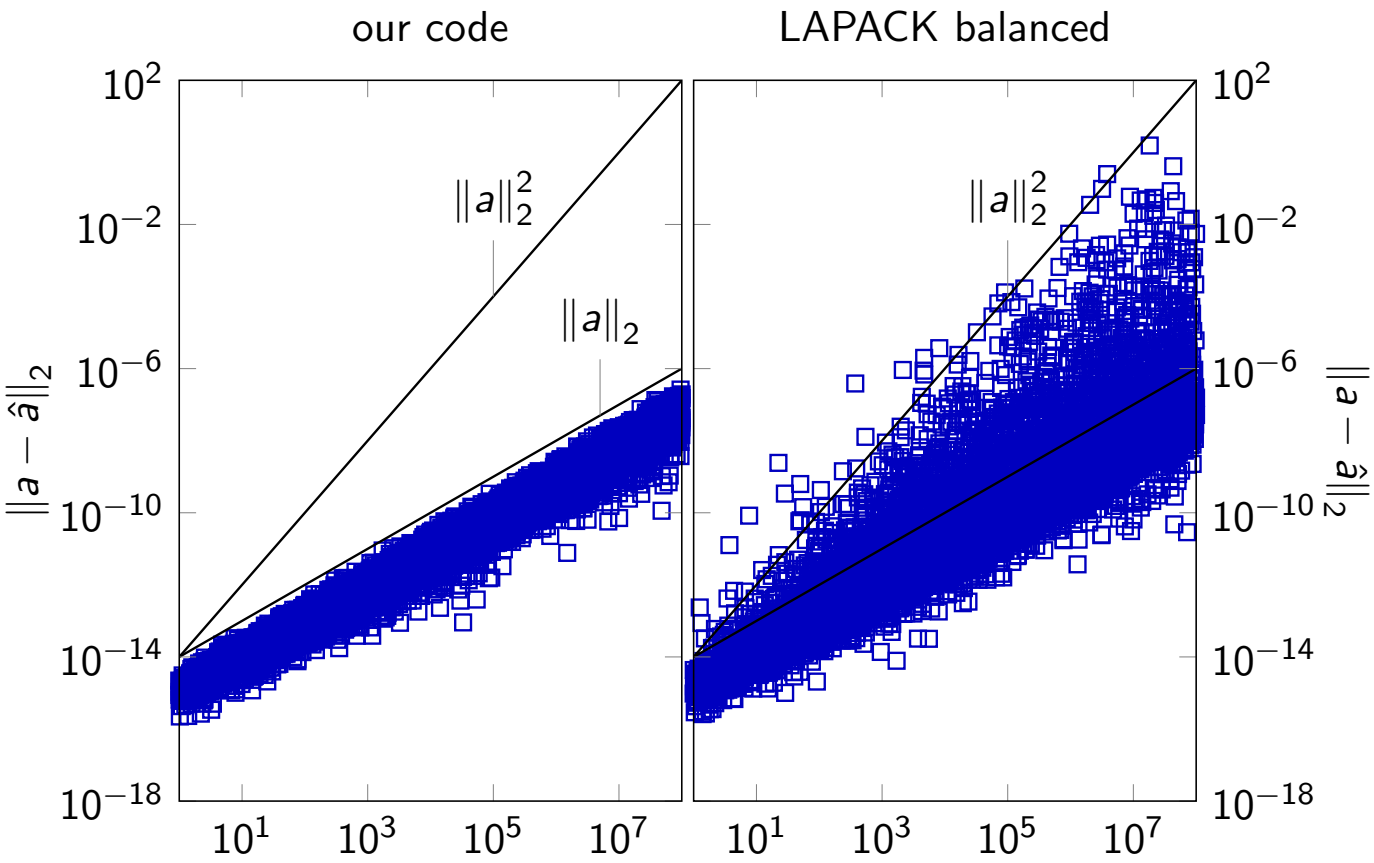

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Thank you for your attention.