Understanding the QR algorithm, Part X

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1. Understanding the QR algorithm, SIAM Rev., 1982

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8. The QR algorithm revisited, SIAM Rev., 2008.

Some names associated with the QR algorithm

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Implicitly Shifted QR algorithm

- How should we understand it? ...view it? ...teach it to our students?

The Standard Approach . . .

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- . . . it is far removed from versions of the QR algorithm that are actually used.

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implicit-Q theorem vs. Krylov subspaces

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  implicit-Q theorem vs. Krylov subspaces

- Introducing Krylov subspaces improves understanding, allows more general results, and prepares students for Krylov subspace methods.

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- compute $p(A) e_1$

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- Perform similarity transform $A \rightarrow Q_0^* A Q_0$.

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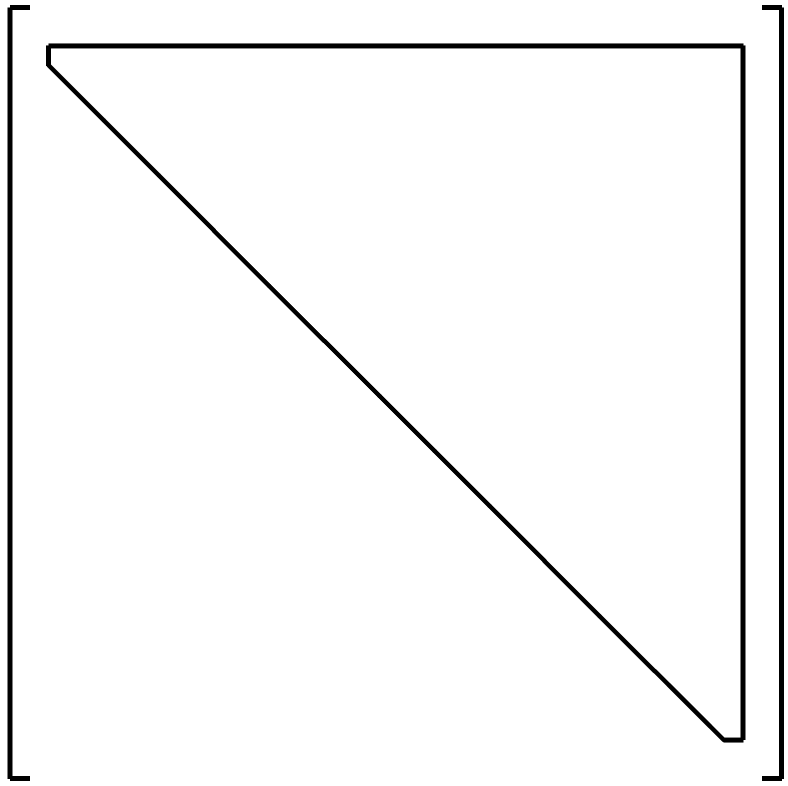
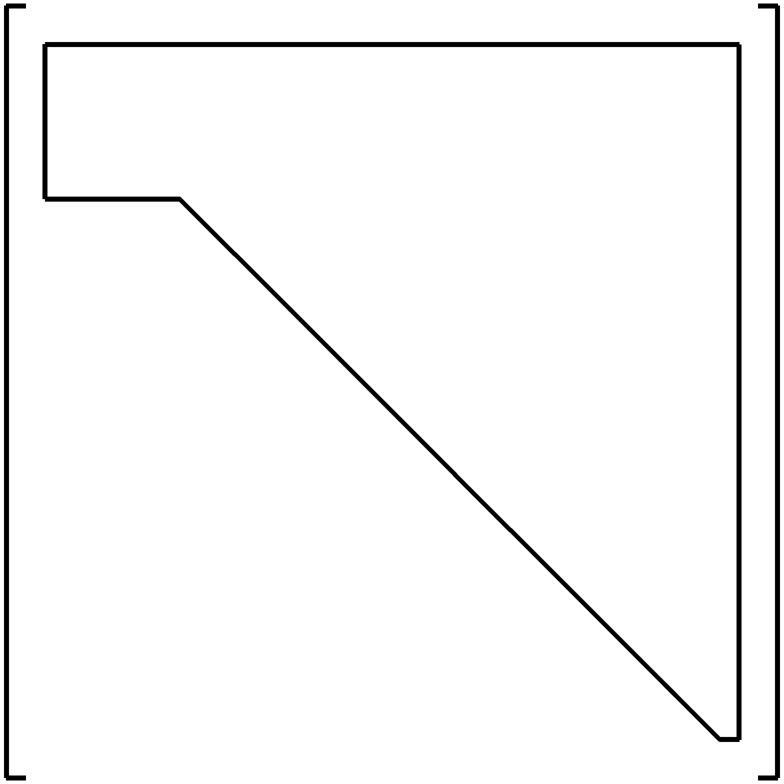
- Perform similarity transform $A \rightarrow Q_0^* A Q_0$.
- Hessenberg form is disturbed.

An Upper Hessenberg Matrix



After the Transformation ($Q_0^* A Q_0$)



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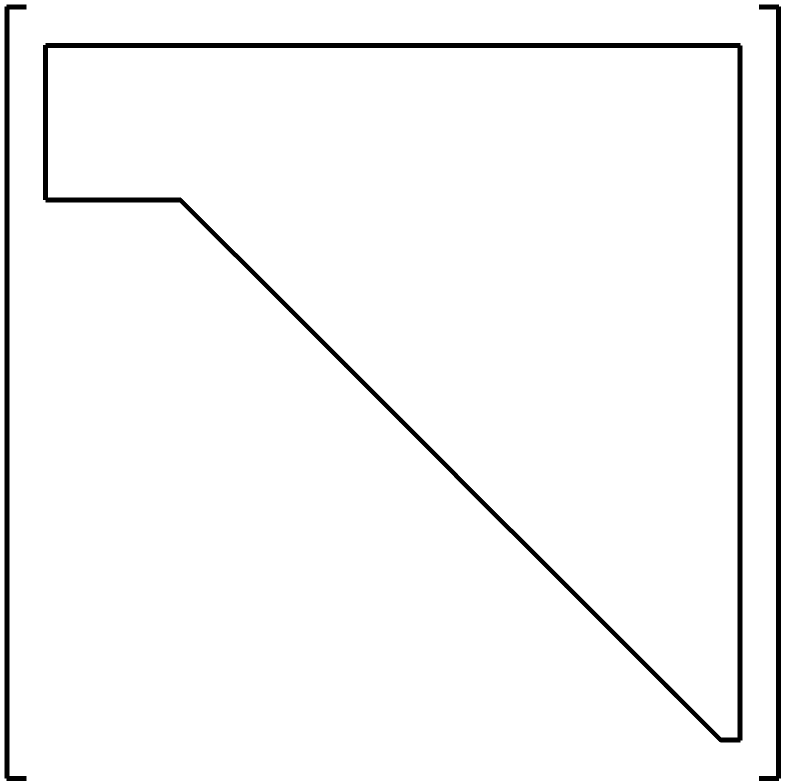
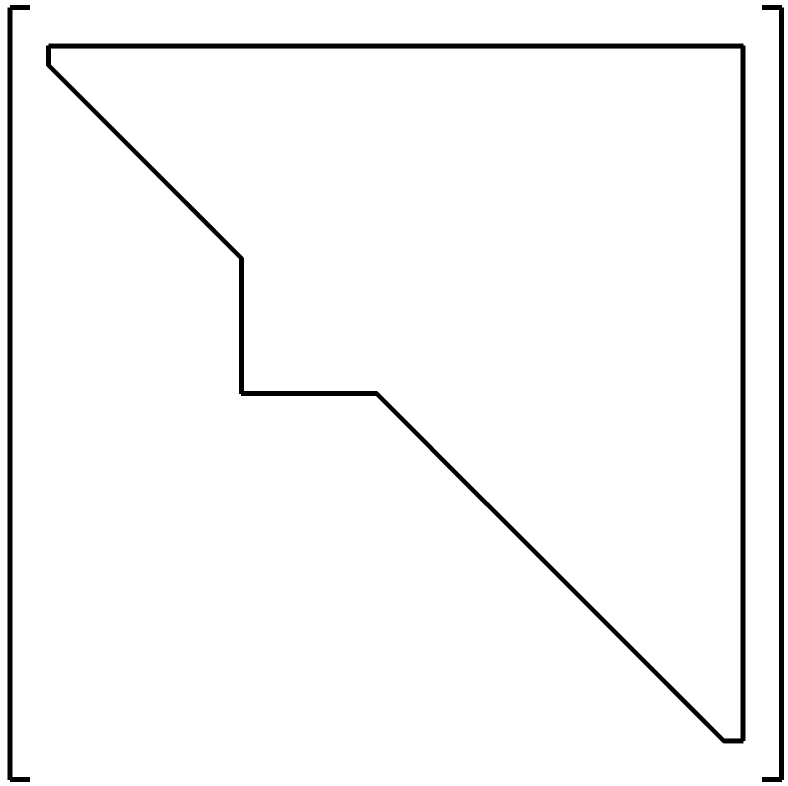


Now return the matrix to Hessenberg form.

Chasing the Bulge



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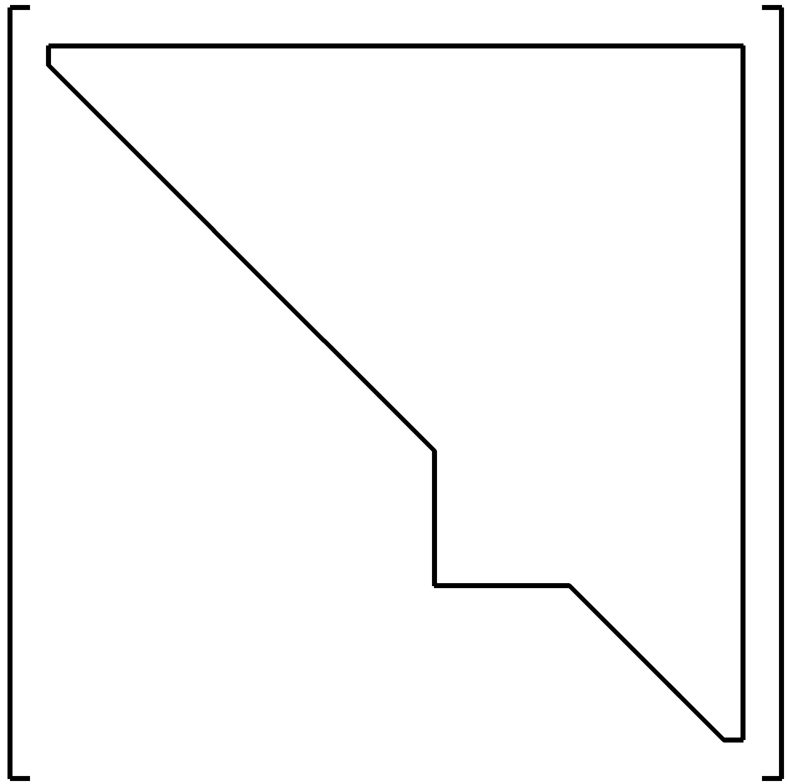
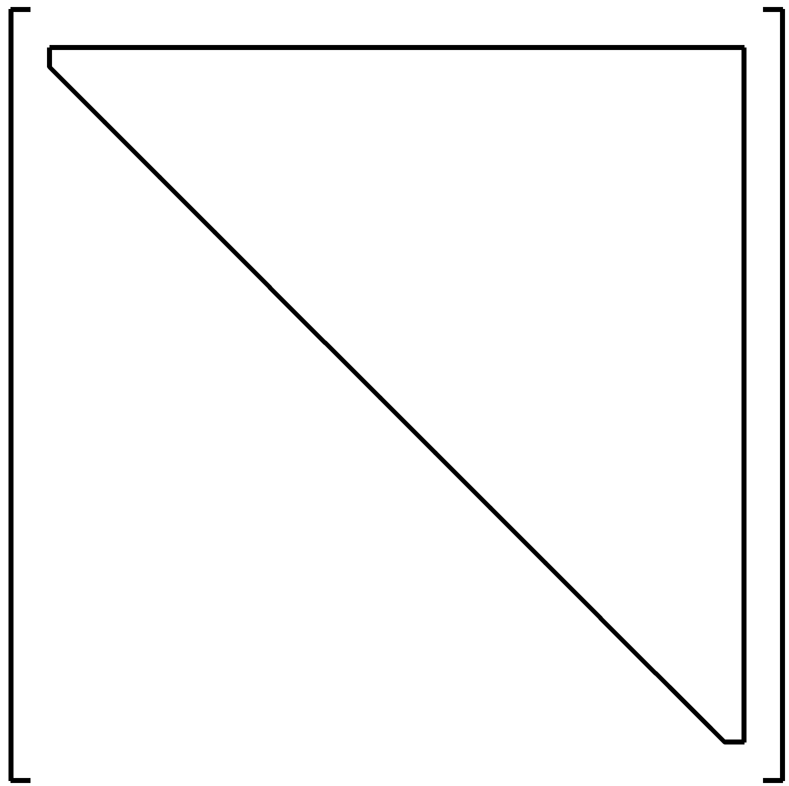


Done



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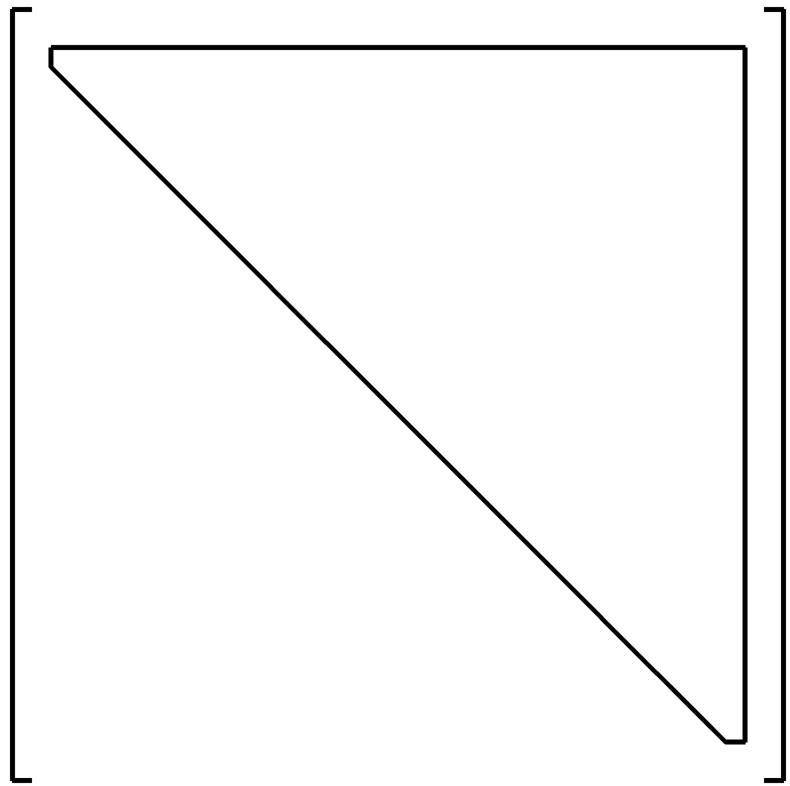

The implicit QR step is complete!

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- $\hat{A} = Q^* A Q$

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- Can we carve a reasonable pedagogical path that leads directly to the implicitly-shifted QR algorithm, bypassing the basic QR algorithm entirely?

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$$A = QR \qquad RQ = \hat{A}$$

- Can we carve a reasonable pedagogical path that leads directly to the implicitly-shifted QR algorithm, bypassing the basic QR algorithm entirely?
- That's what we are going to do today.

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  (instead of implicit-Q theorem)

No Magic Shortcut!

Power Method, Subspace Iteration

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Krylov Subspaces . . .

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- Conclusion: Power method induces nested subspace iterations on Krylov subspaces.

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- convergence rates:

$$|\, p(\lambda_{j+1})/p(\lambda_j)\,|, \quad j = 1,\ 2,\ 3, \ldots$$

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- triangular form (eigenvalues)
- relationship of invariant subspaces to triangular form

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with change of coordinate system

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Subspace Iteration with change of coordinate system

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- $$\begin{aligned} p(A)\mathcal{S} &= \text{span}\{p(A)e_1, \ldots, p(A)e_j\} \\ &= \text{span}\{q_1, \ldots, q_j\} \text{ (orthonormal)} \end{aligned}$$

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- $\text{span}\{q_1, \ldots, q_j\} \longrightarrow \text{span}\{e_1, \ldots, e_j\}$
- ready for next iteration

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- . . . holds the subspace fixed

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- while the matrix changes.

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- $A \rightarrow \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ (A_{11} is $j \times j$.)

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Reduction to Hessenberg form

- $Q \to Q^*AQ = H$ (a similarity transformation)
- can always be done (direct method, $O(n^3)$ flops)
- brings us closer to triangular form

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- Example: $q_1 = \alpha p(A) e_1$

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- More generally . . .

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Proof (sketch): Induction on j. $AQ = QH$

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- Transform A to upper Hessenberg form:

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by a matrix Q that has $q_1 = \alpha p(A) e_1$.

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- ...how to get BLAS 3 speed?
- ...how to parallelize?

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- The implicit-Q theorem is avoided.
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- Krylov subspace methods are foreshadowed.

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- Thank you for your attention.