Dealing with Hamiltonian Structure: Challenges and Successes

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Alternating Pencils

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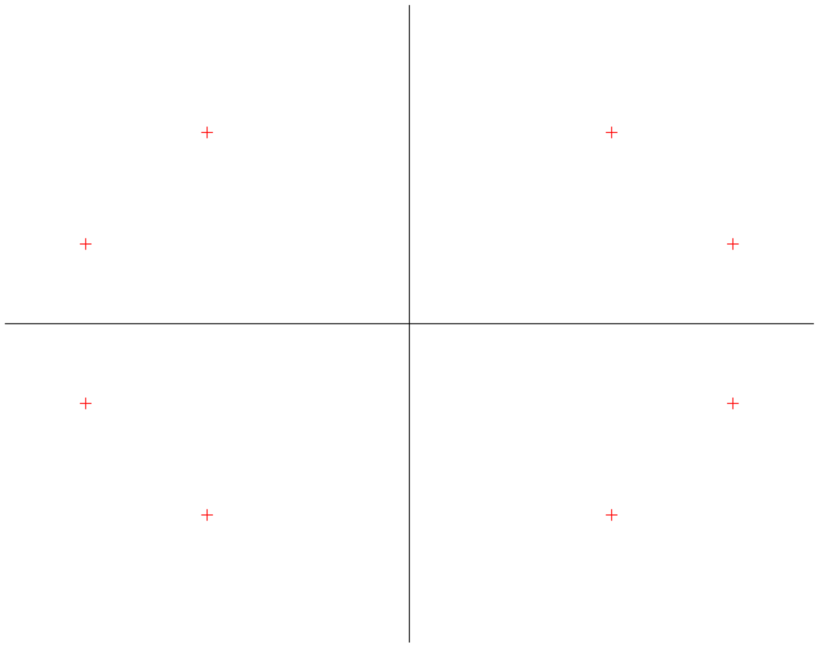
- symmetry of spectrum

Spectrum of an Alternating Pencil



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- Mackey/Mackey/Mehl/Mehrmann

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- Hamiltonian matrix $\Leftrightarrow$ alternating pencil

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- Deflate. (many details skipped)

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