Core-Chasing Algorithms for Eigenvalue Computation

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Department of Mathematics
Washington State University

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Today's Topic

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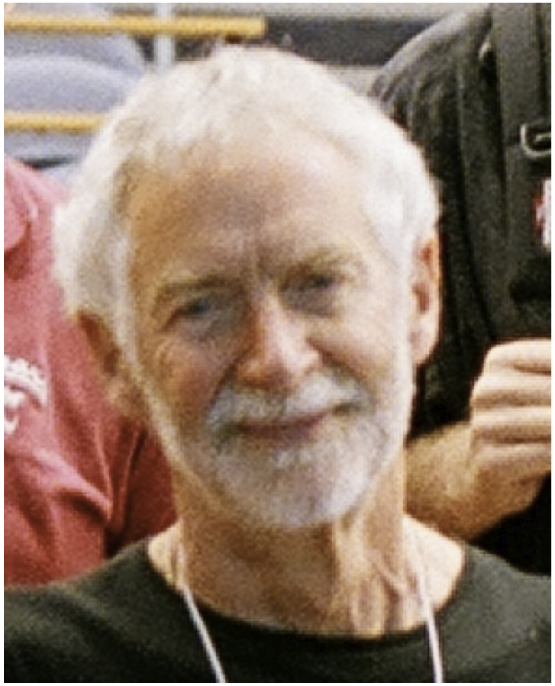
- Find the eigenvalues (. . . vectors, invariant subspaces)
- Many applications
- Interest dates back to the very beginning of the electronic computing era.
- Nobody knew how to do it.

John Francis



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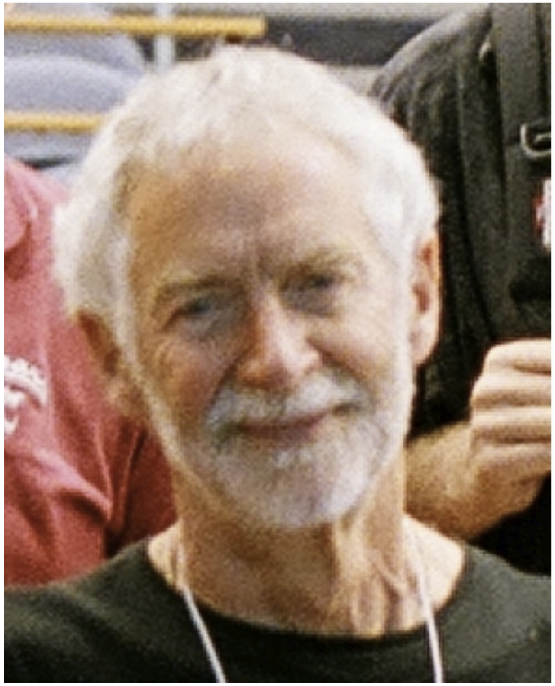

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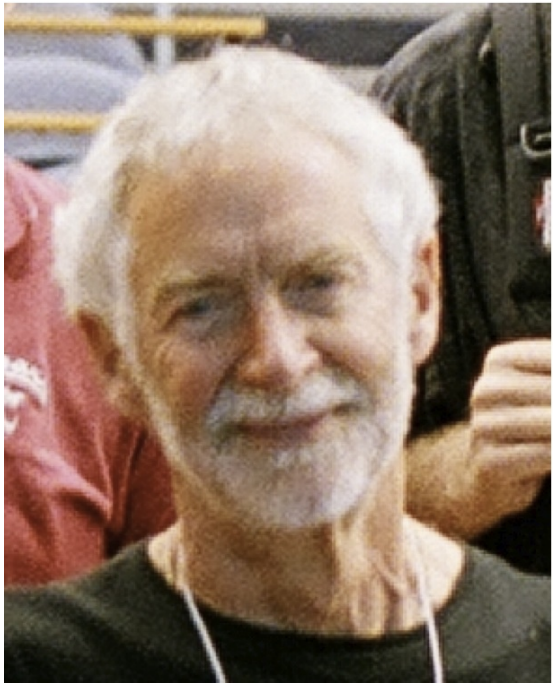

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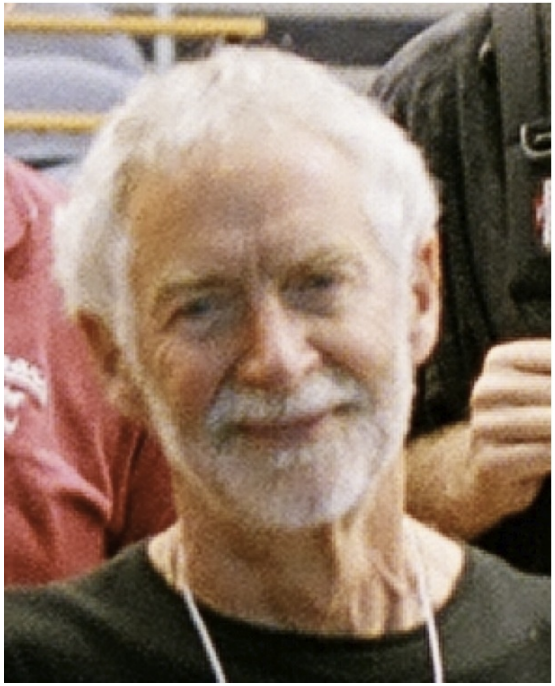

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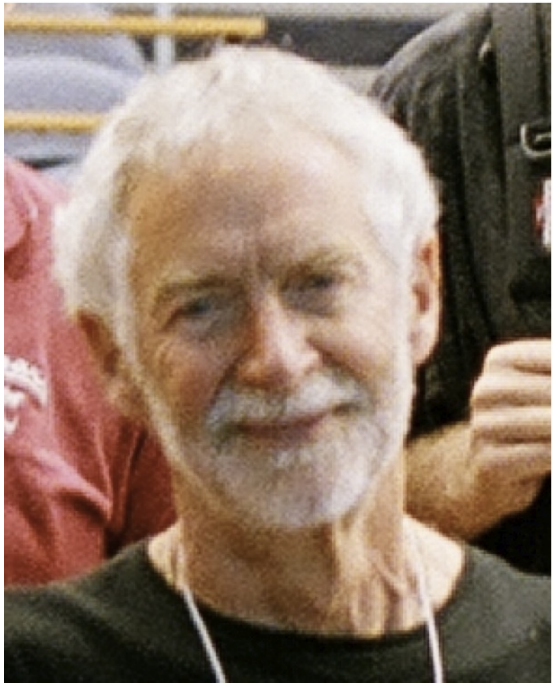

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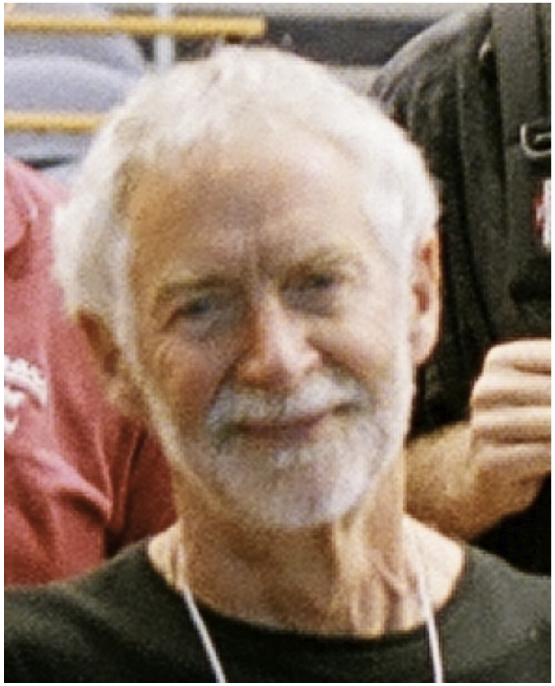

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Francis's algorithm (superficial description)

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- upper Hessenberg form

$$A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & * & * & * & * \\ & & * & * & * \\ & & & * & * \end{bmatrix}$$

- unitary similarity transformation
- direct method ($O(n^3)$ flops)

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- (Galois theory)

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Francis's algorithm (superficial description)

Chasing the bulge

- iteration complete!
- repeated iterations $\Rightarrow$ triangular form
- This is the *single-shift* algorithm.
- *Double-shift* algorithm chases a bigger bulge.

Computational Cost

- $O(n^2)$ flops per iteration
- $O(n)$ total iterations
- $O(n^3)$ total flops

For details see . . .

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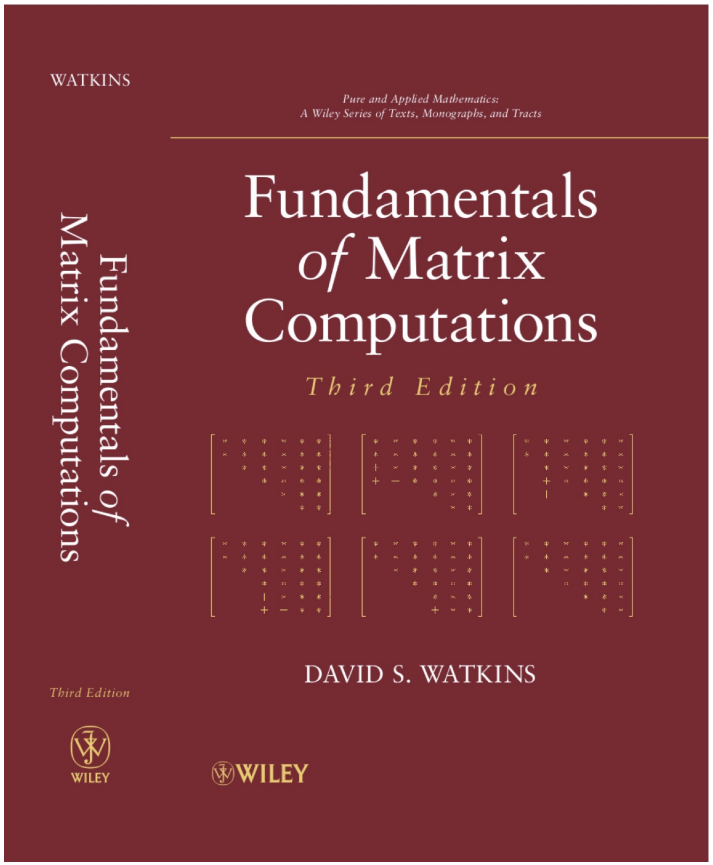

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. . . but we're still not done!

Our International Research Group

This is joint work with

- **Jared Aurentz** (Oxford)
- **Thomas Mach** (KU Leuven)
- **Raf Vandebril** (KU Leuven)

A new look at an old algorithm

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Store in QR decomposed form

$$A = QR$$

- Q is unitary, R is upper triangular

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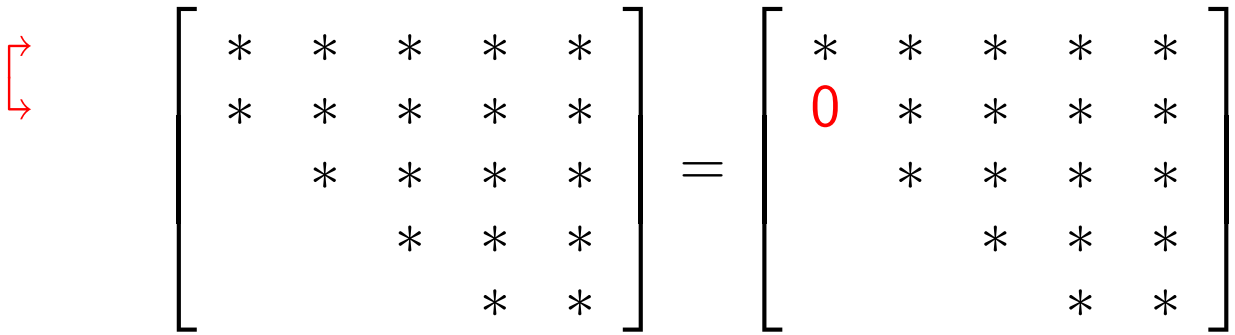

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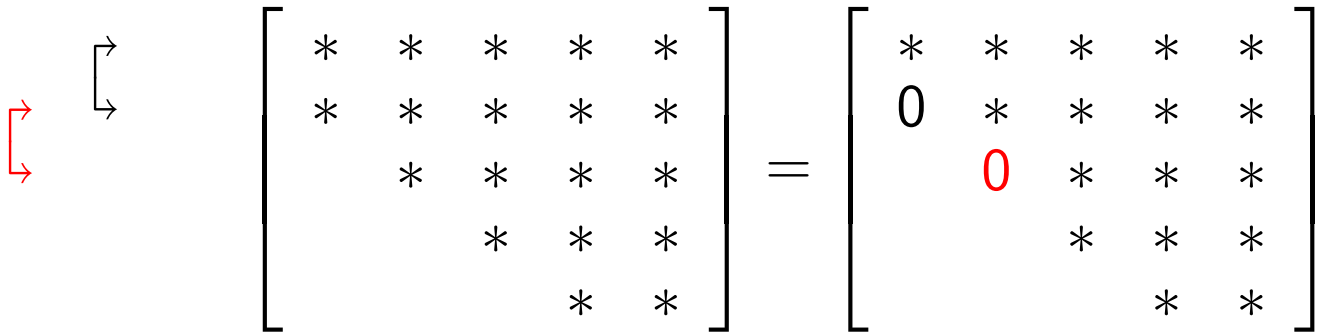

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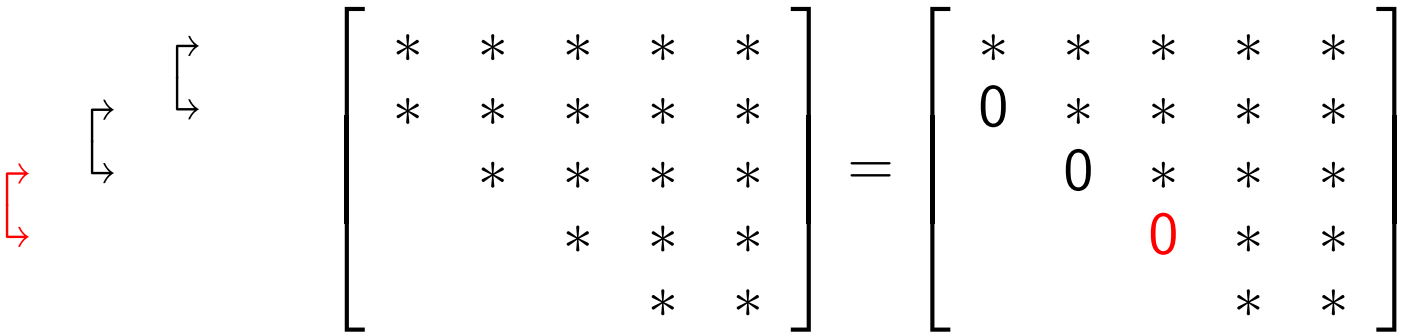

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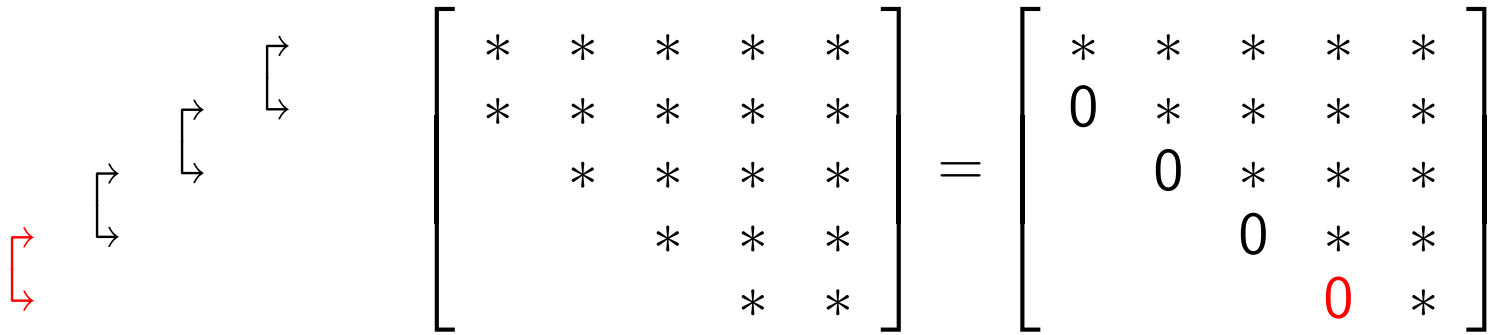

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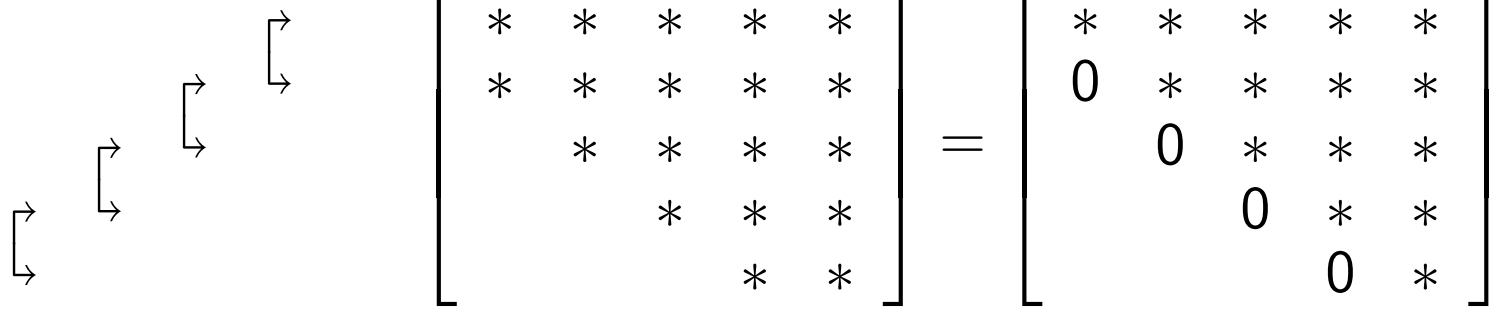

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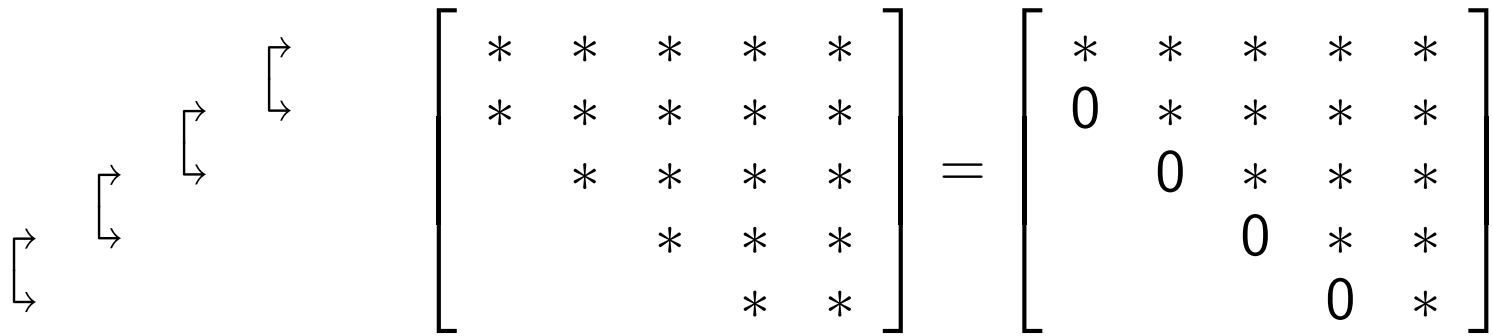

- **Def:** *Core Transformation*

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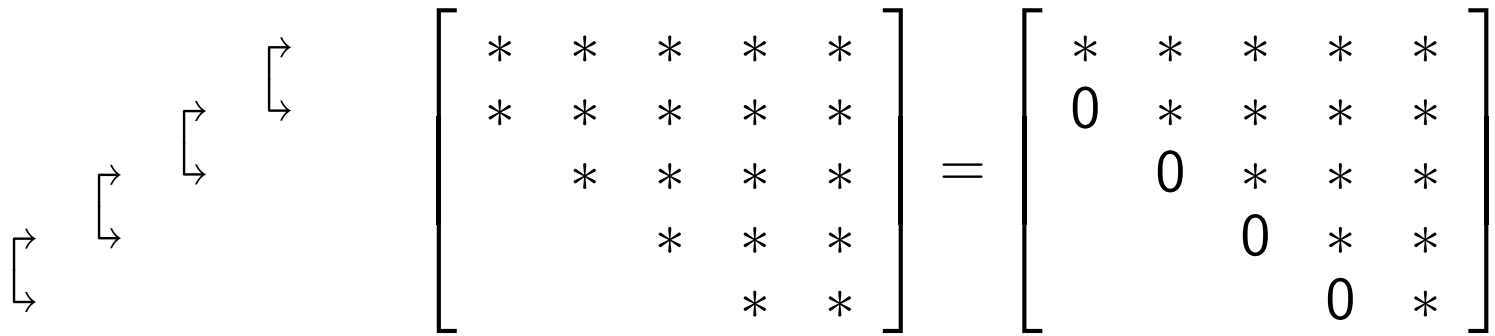

- **Def:** *Core Transformation*
- Now invert the core transformations to move them to the other side.

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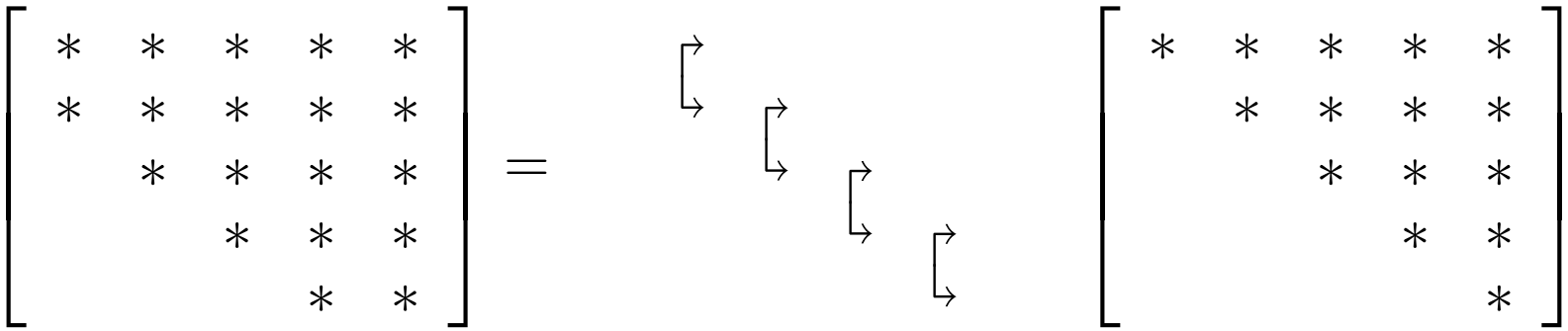

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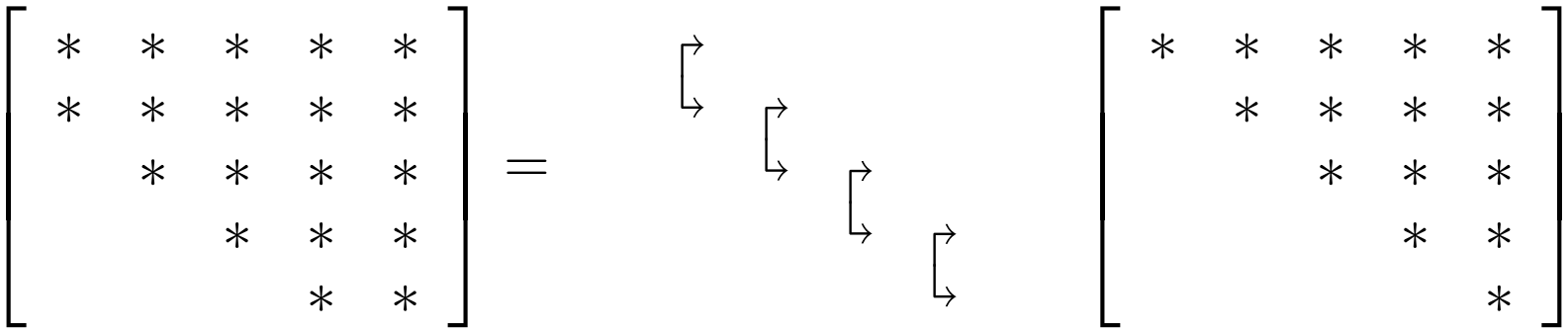

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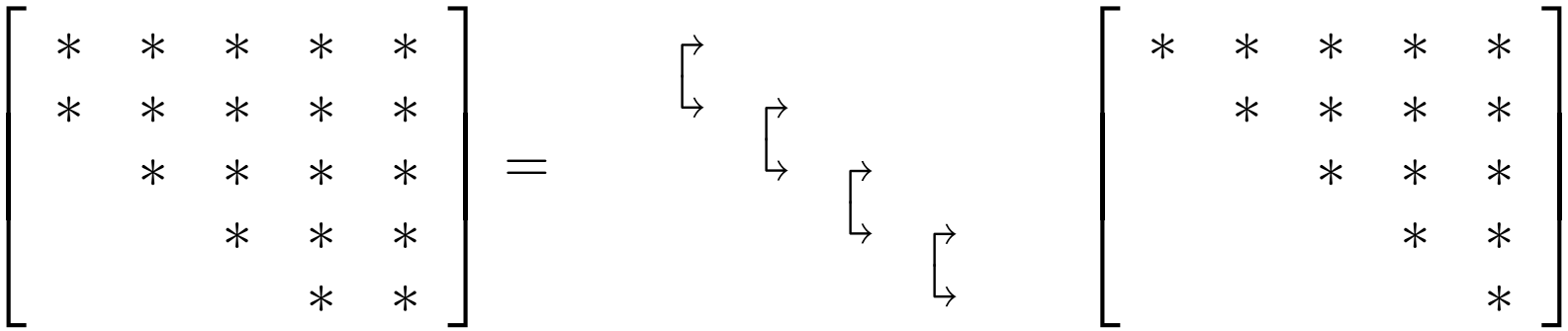

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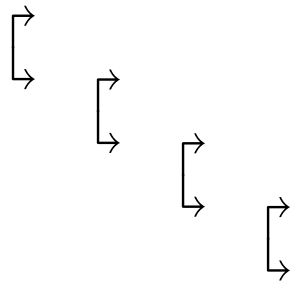
$$Q = \quad$$ 

Q requires only $O(n)$ storage space.

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Manipulating core transformations

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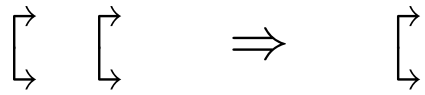
- Fusion



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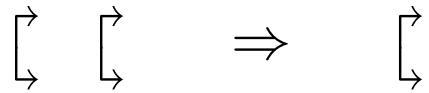
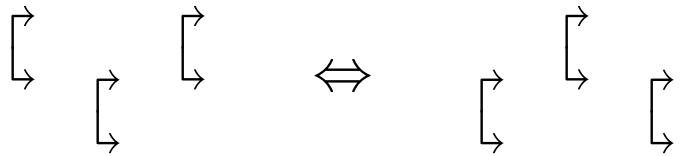


- Turnover (aka shift through, Givens swap, . . .)



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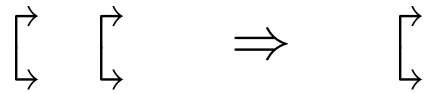

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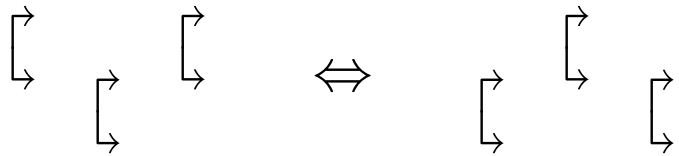
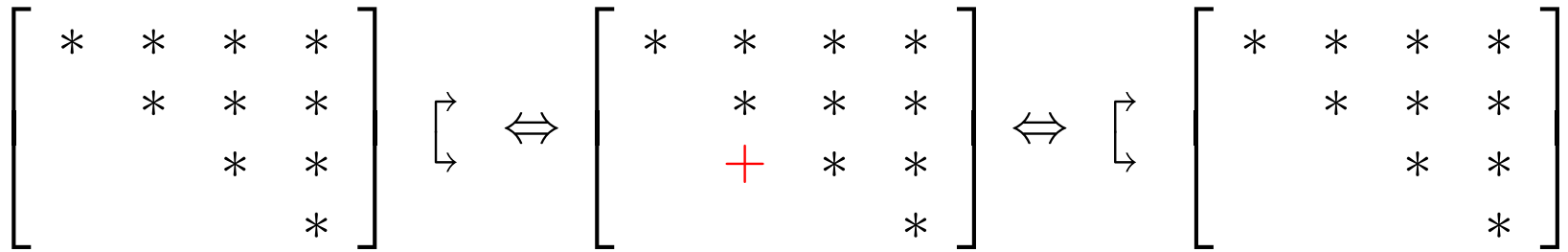


- Passing a core transformation through a triangular matrix (cost $O(n)$)



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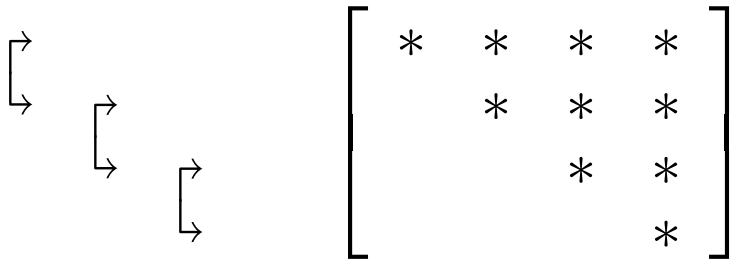
Francis's algorithm on the QR decomposed form (a core chasing algorithm)



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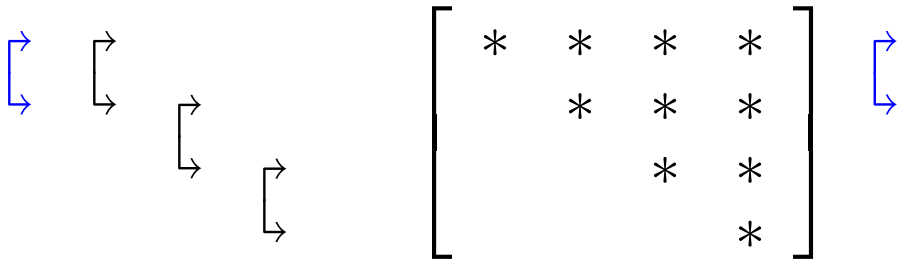

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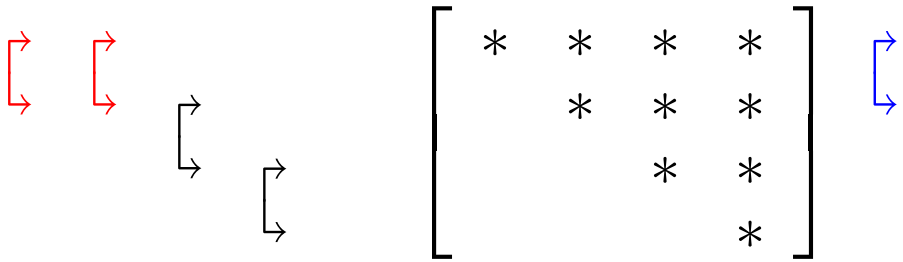

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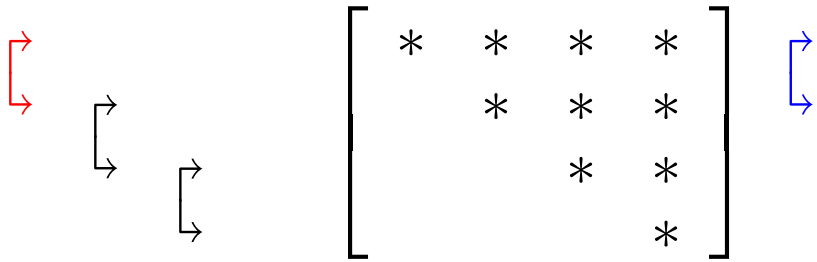

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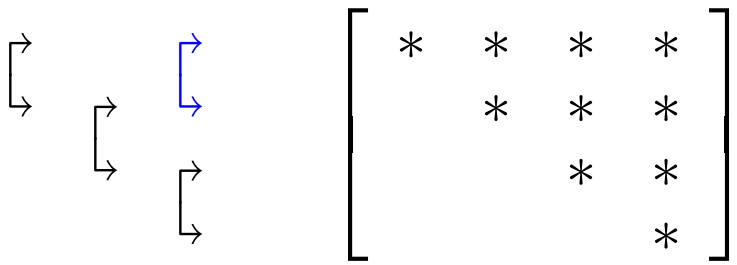

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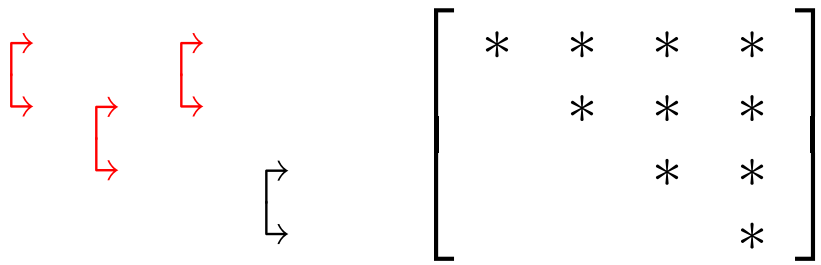

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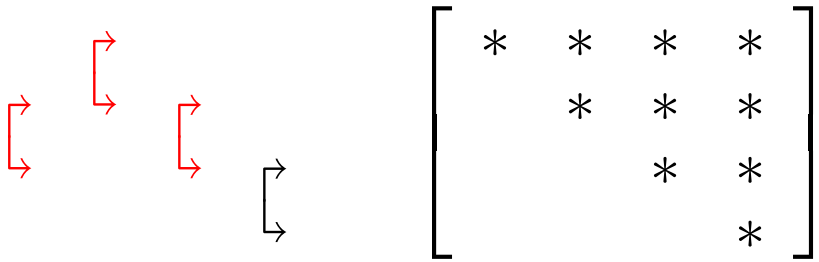

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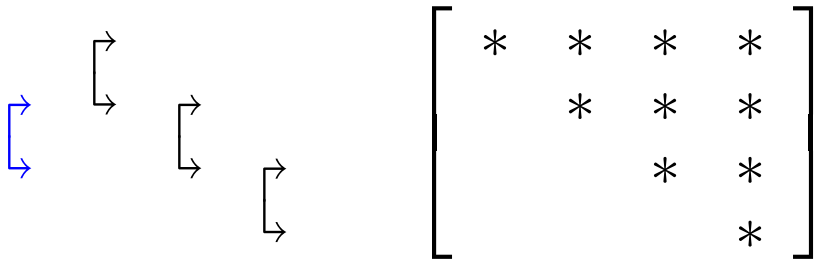

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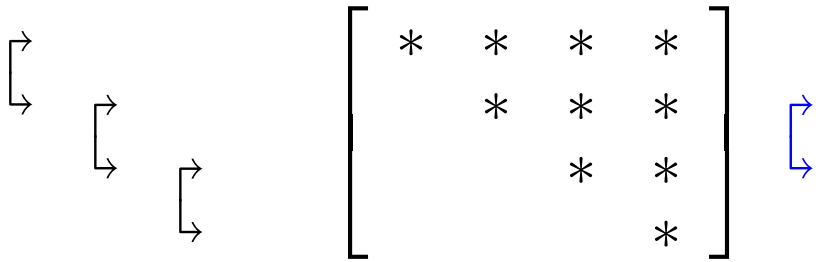

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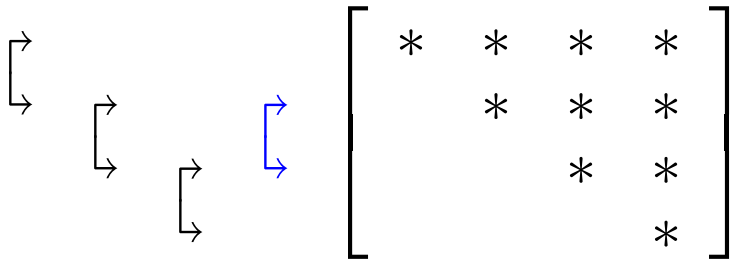

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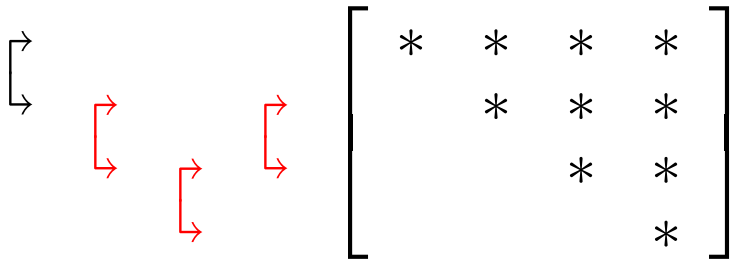

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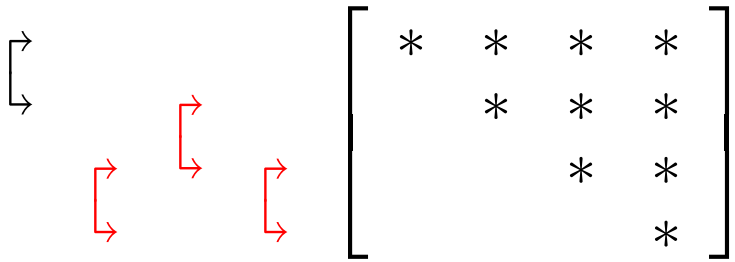

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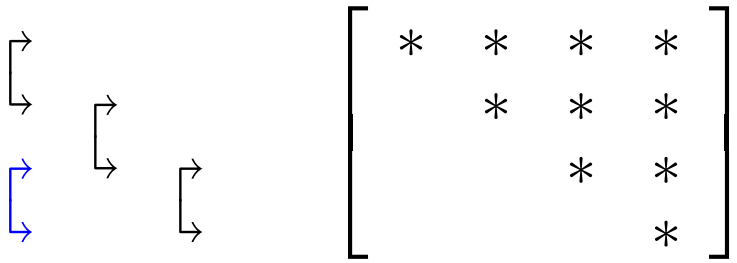

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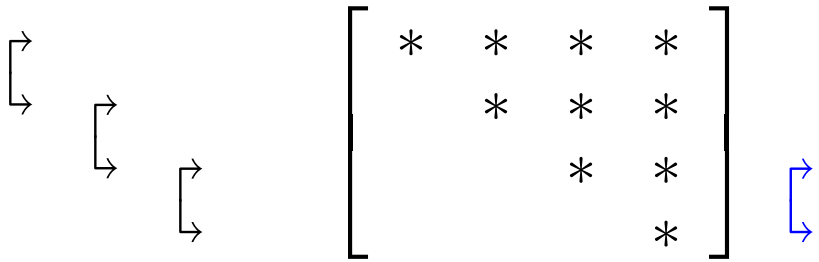

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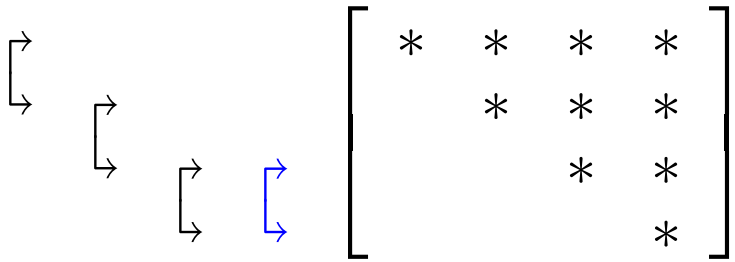

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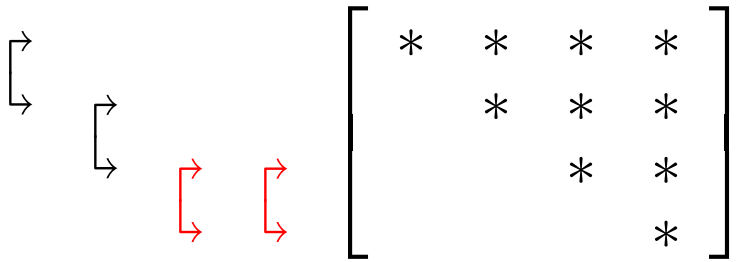

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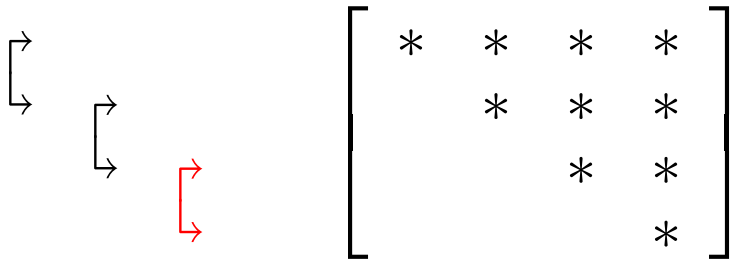

A new look at an old algorithm

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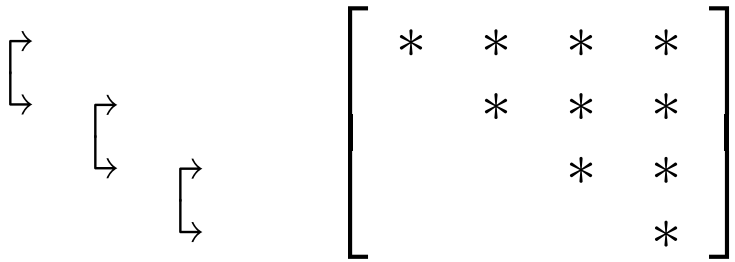

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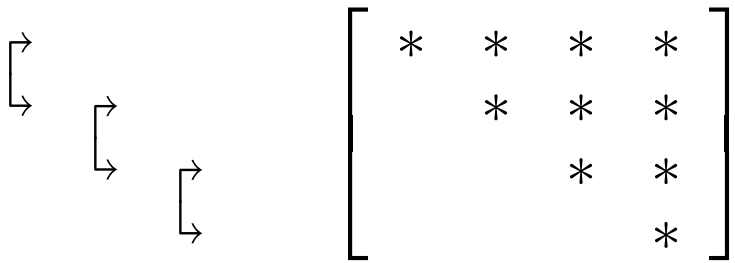

Done!

Cost

Cost

- Most arithmetic in passing-through operation

Cost

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- $O(n^2)$ flops per iteration . . .
- $O(n^3)$ total flops . . .
- about the same as for standard Francis iteration.

Are there any advantages?

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- unitary case

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- unitary case
- companion case (unitary-plus-rank-one)

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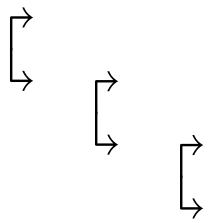
- unitary case
- companion case (unitary-plus-rank-one)
- general case: efficient cache use

Unitary Case

Unitary Case

$$A = QR = \begin{bmatrix} \curvearrowright & & \\ & \curvearrowright & \\ & & \curvearrowright \end{bmatrix} \begin{bmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{bmatrix}$$

Unitary Case

$A = QR =$ 

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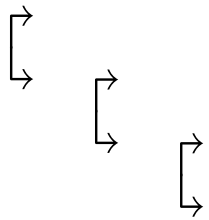
$A = QR =$ 

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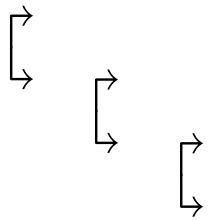
$A = QR =$ 

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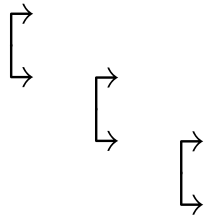
$A = QR =$ 

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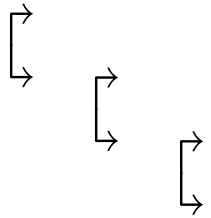
Unitary Case

$A = QR =$ 

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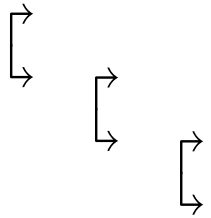
$A = QR =$ 

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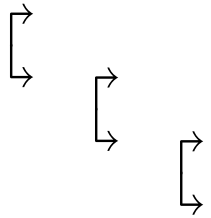
$A = QR =$ 

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- Ammar, Reichel, M. Stewart, Bunse-Gerstner, Elsner, He, W, ...

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$$A = \begin{bmatrix} 0 & & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ & 1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{bmatrix}$$

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Some of the Competitors

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- We will do something else.
- Our method is faster, and we can prove backward stability.

Structure

Structure

- Companion matrix is unitary-plus-rank-one

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 & -a_0 - 1 \\ 0 & & 0 & -a_1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & -a_{n-1} \end{bmatrix}$$

- We exploit this structure.

. . . but we store the QR decomposed form

$$A = QR$$

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$$A = QR$$

$$= \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & -a_1 \\ & 1 & & -a_2 \\ & & \ddots & \vdots \\ & & & -a_0 \end{bmatrix}$$

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- Next step: Roll up x.

Representation of R

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Representation of R

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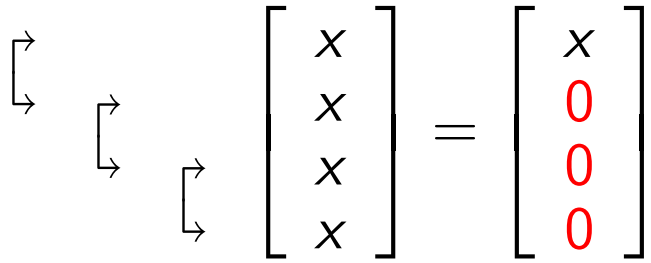
Representation of R



Representation of R

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$$C_1 \cdots C_{n-1} C_n x = \alpha e_1 \qquad \text{(w.l.g. } \alpha = 1\text{)}$$

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- $O(n)$ storage
- Bonus: Redundancy! No need to keep track of y.

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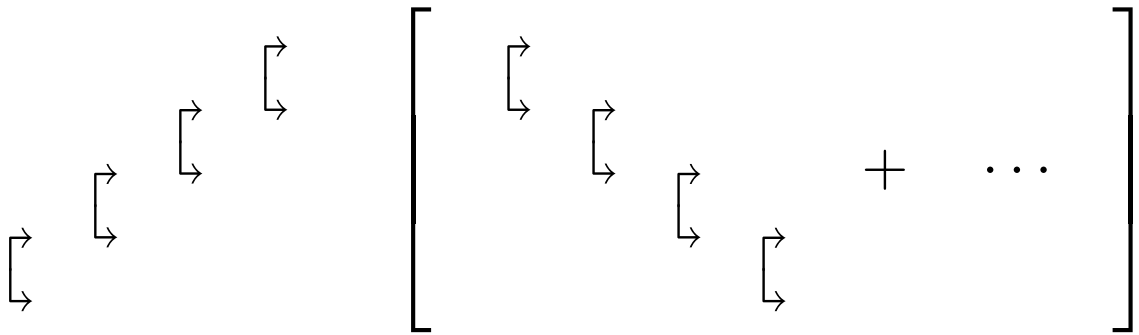
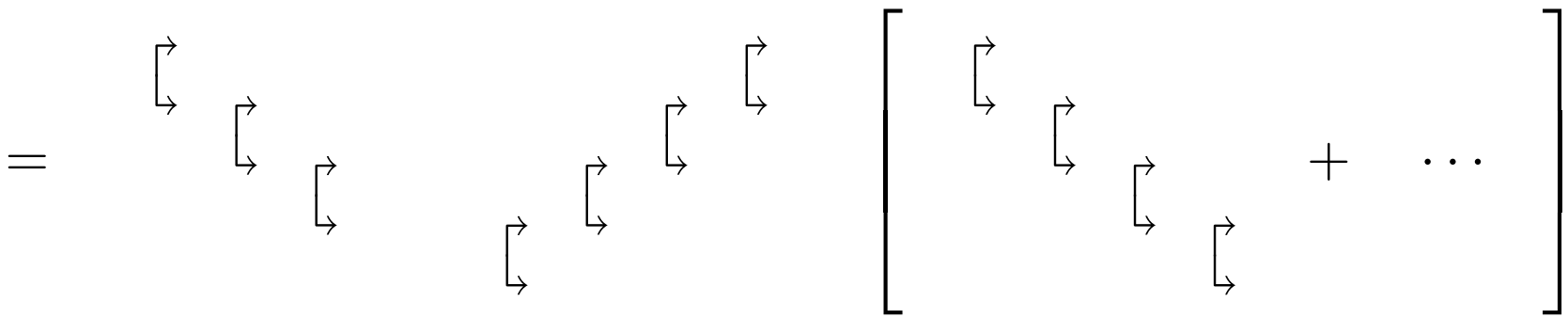
$$=$$



Altogether we have

$$A = QR$$



Francis Iterations

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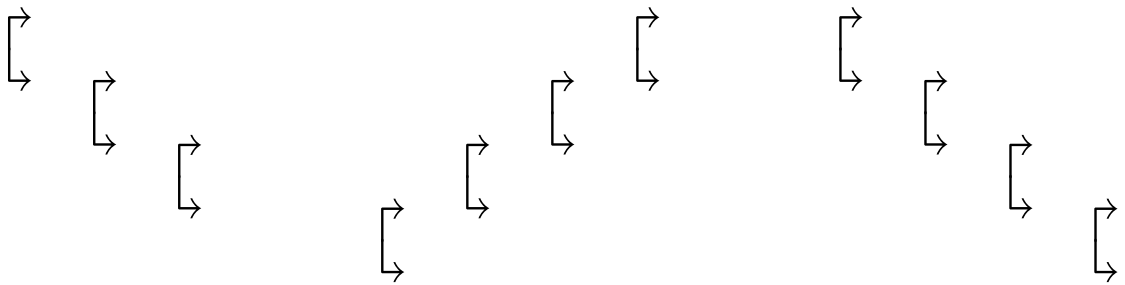
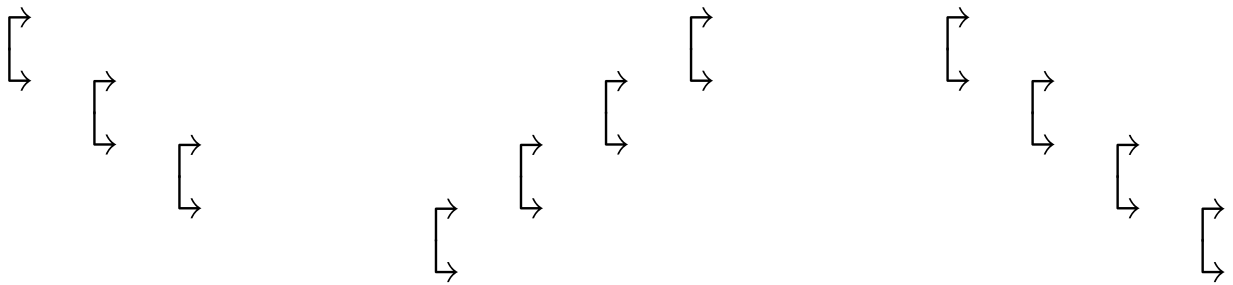
$A =$ 

The Core Chase



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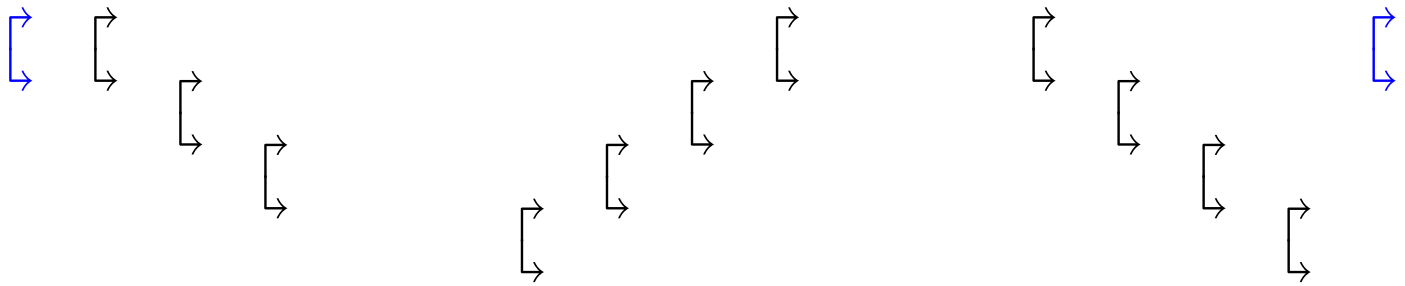

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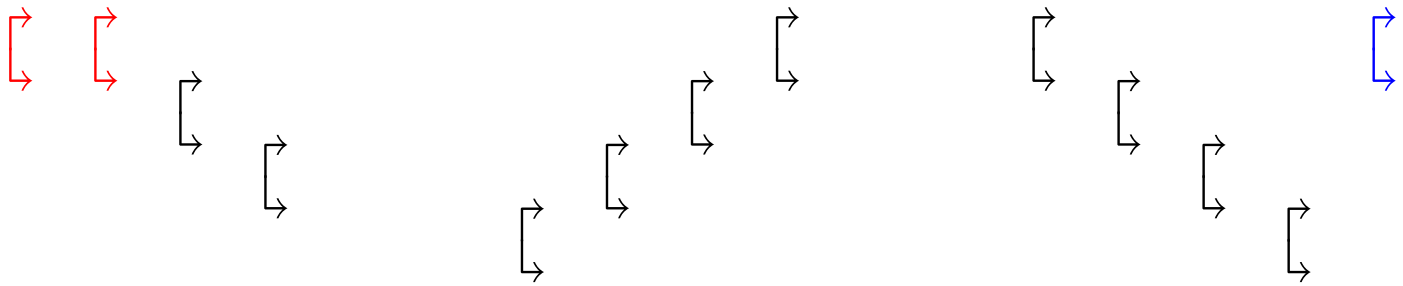

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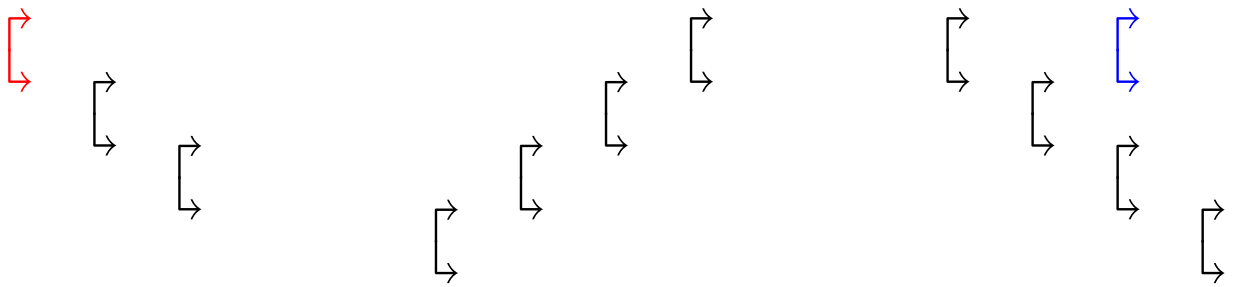

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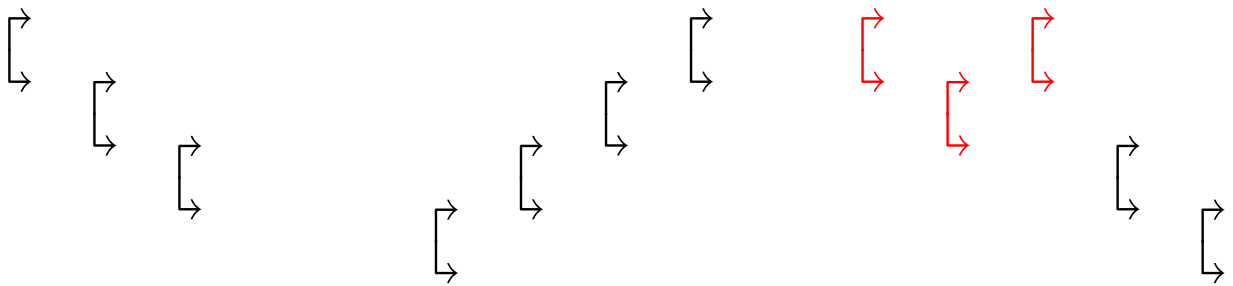

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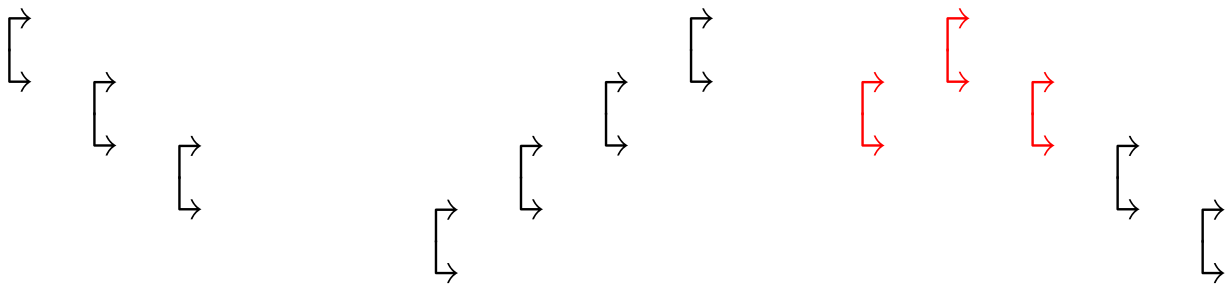

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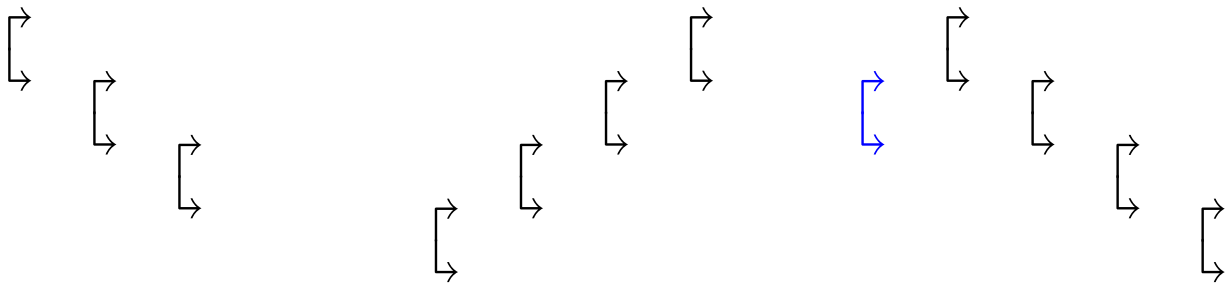

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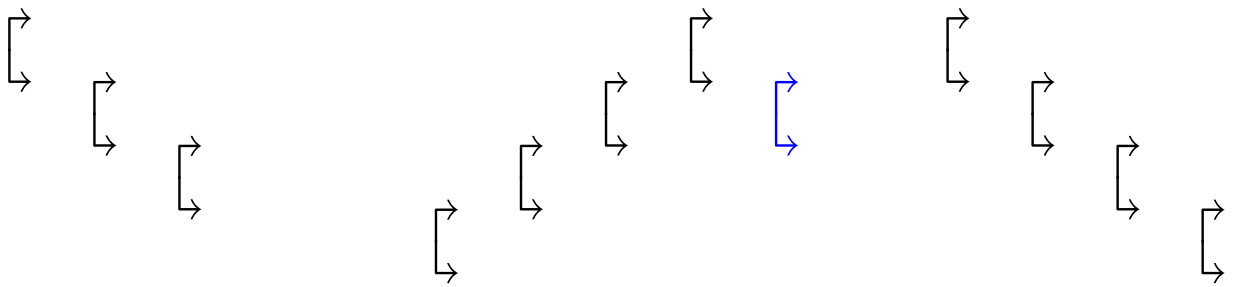

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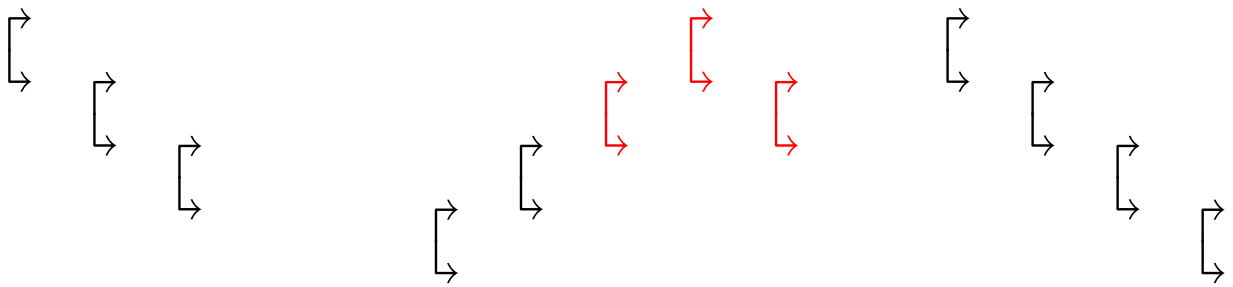

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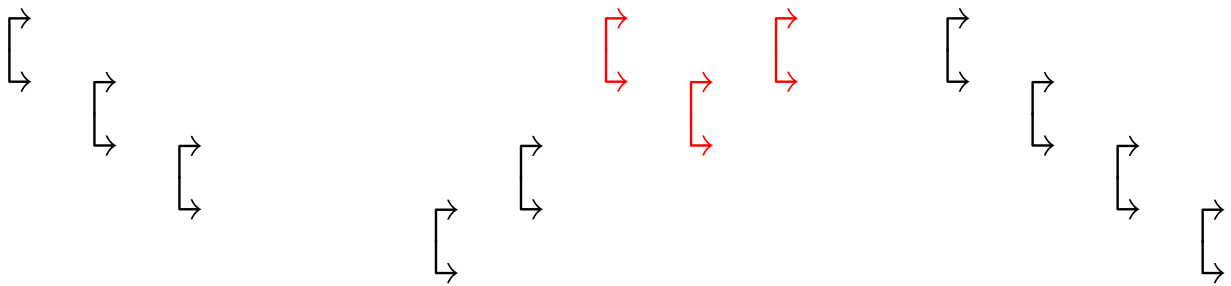

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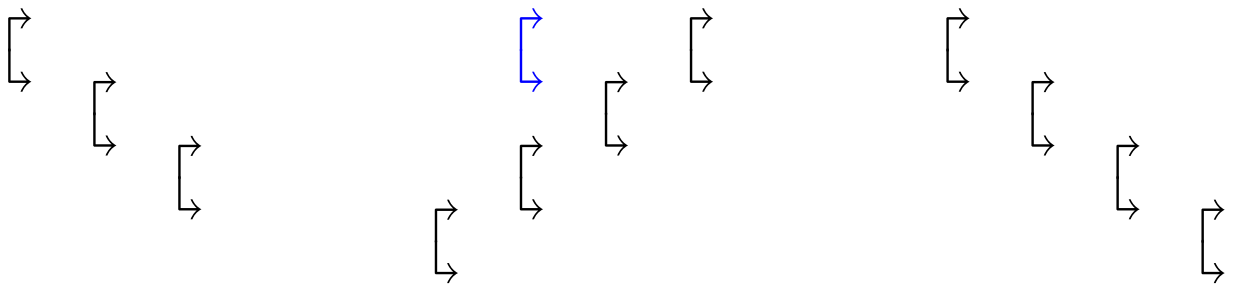

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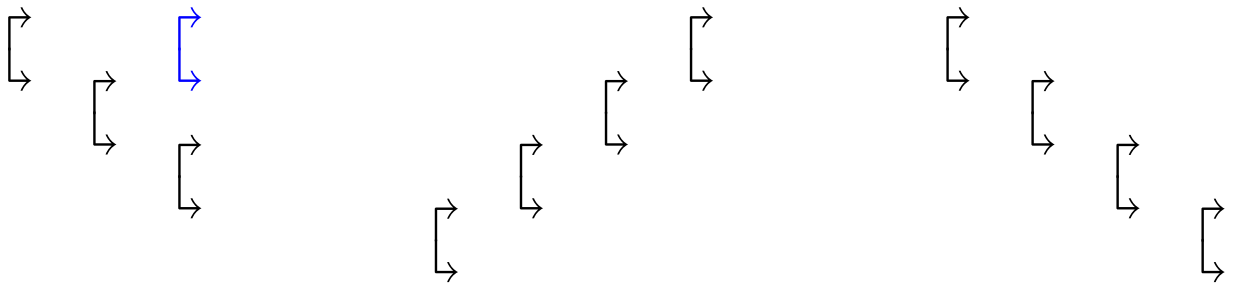
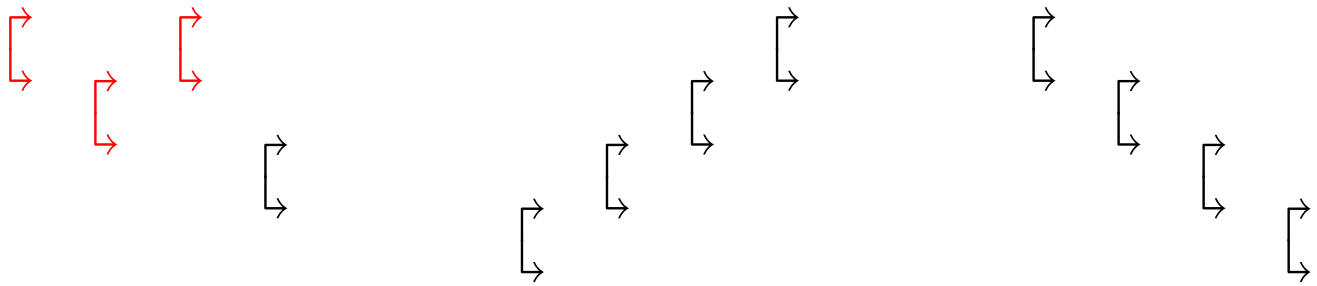


The Core Chase



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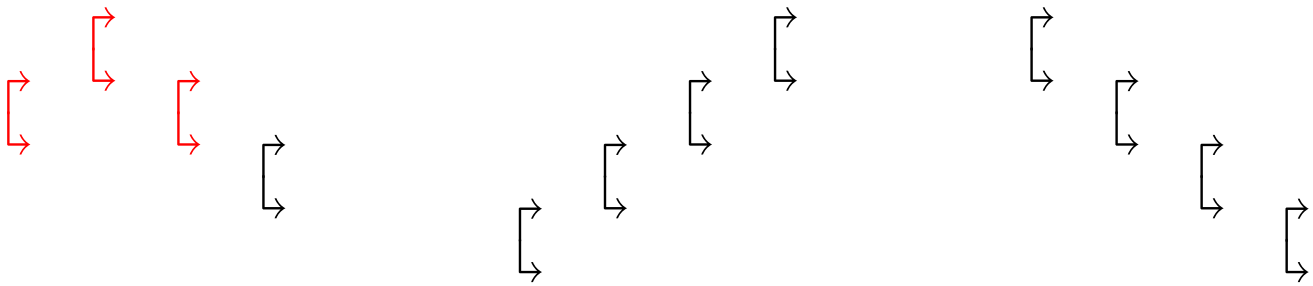

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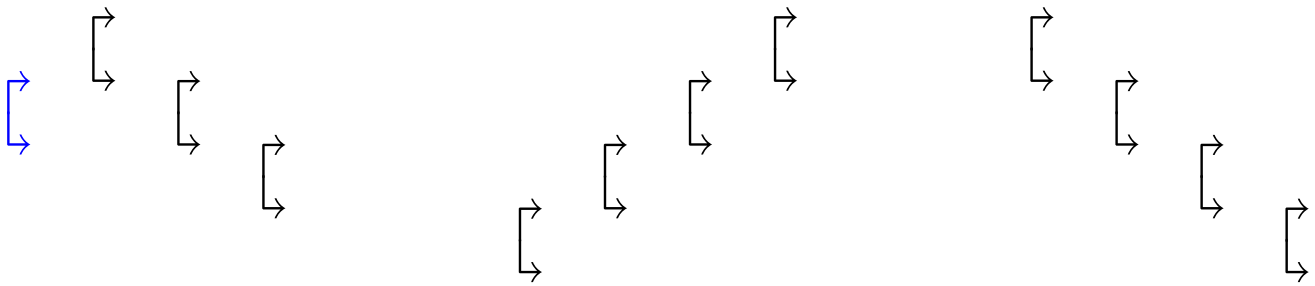

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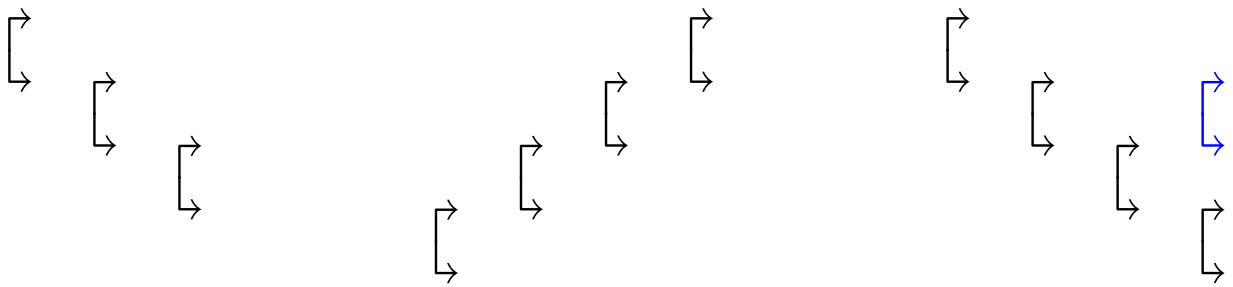

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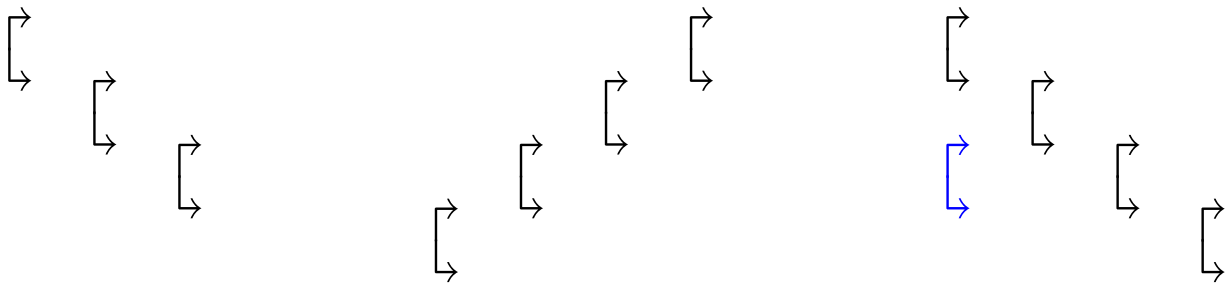

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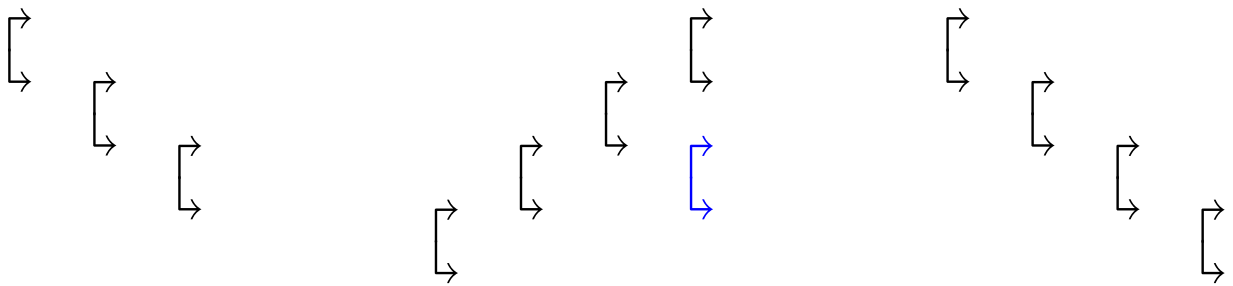

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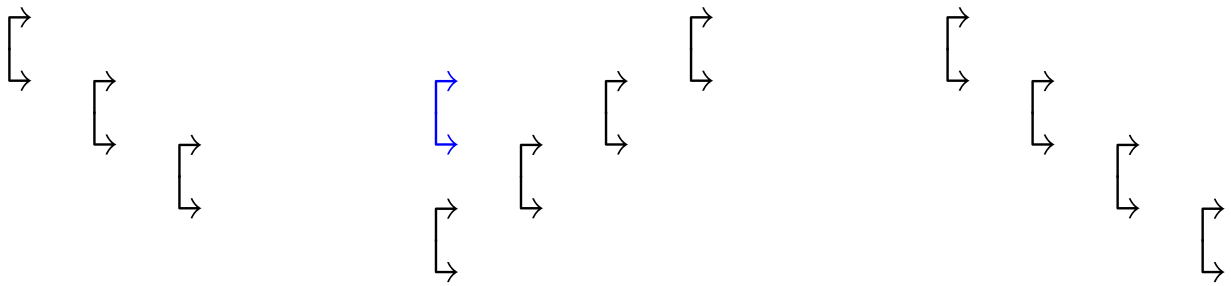

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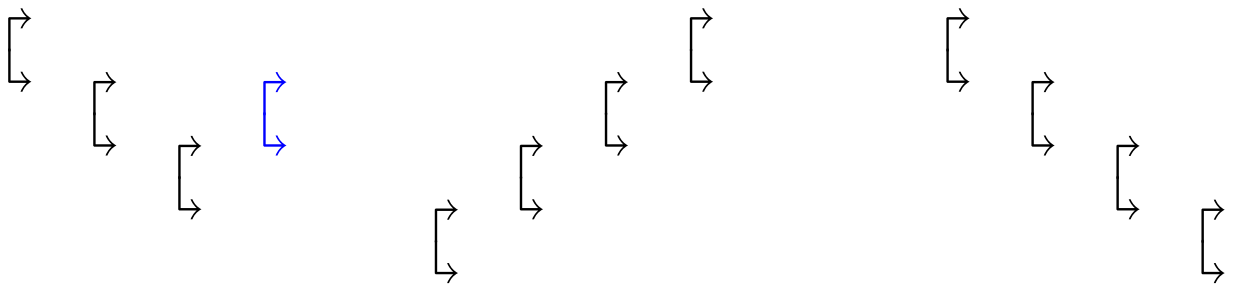

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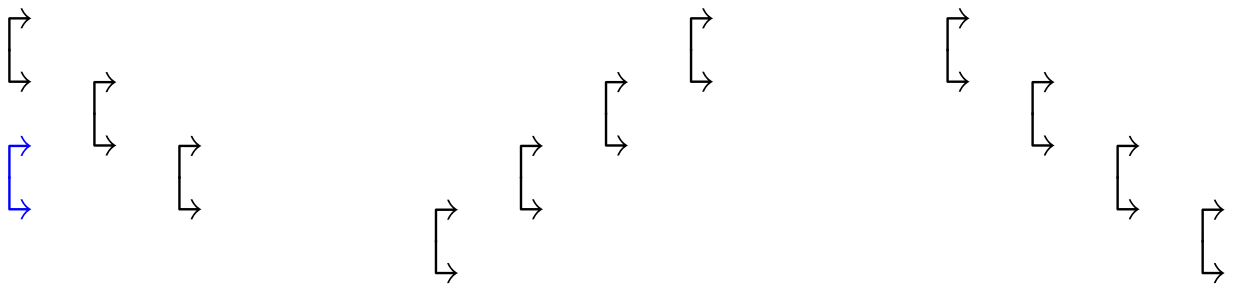

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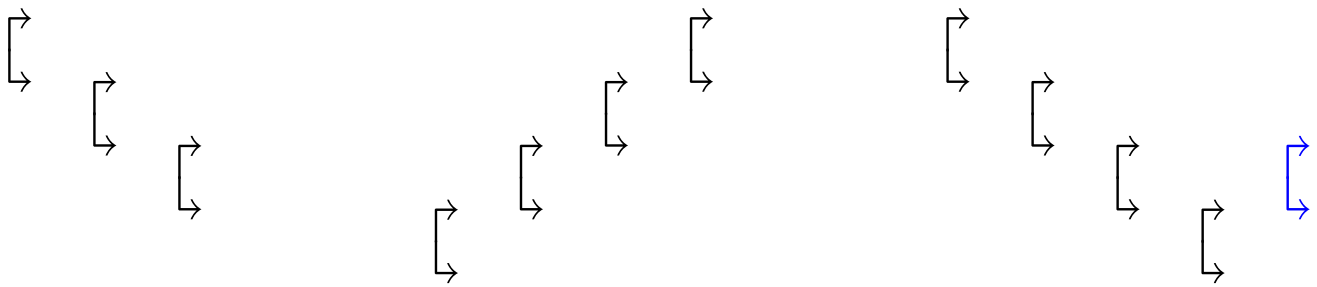

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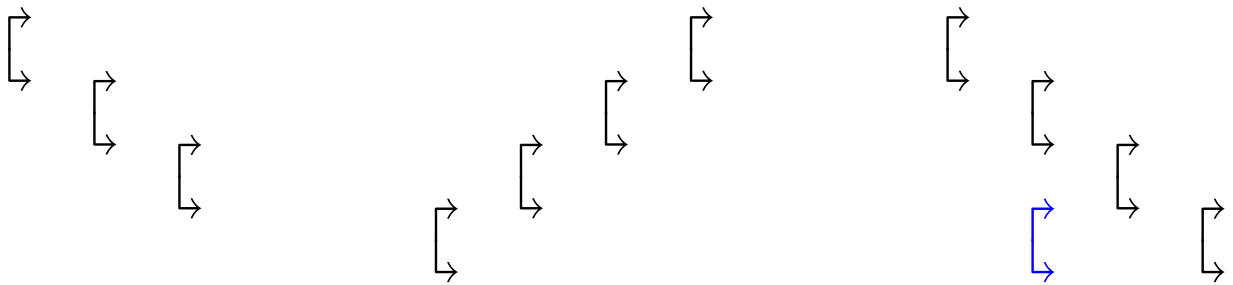

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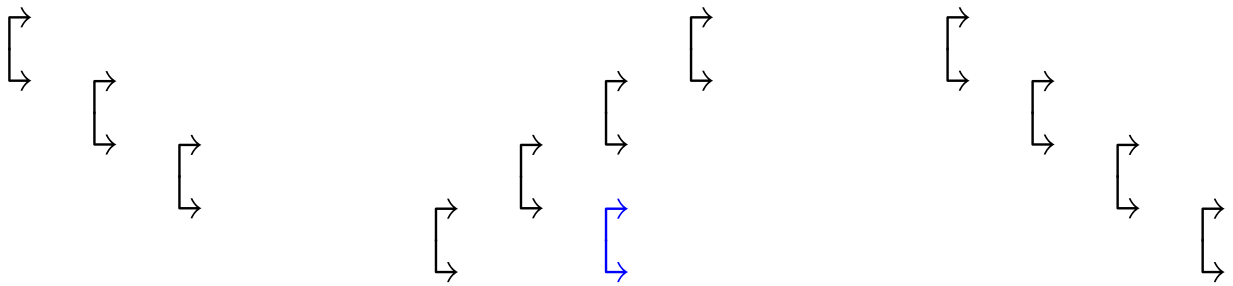

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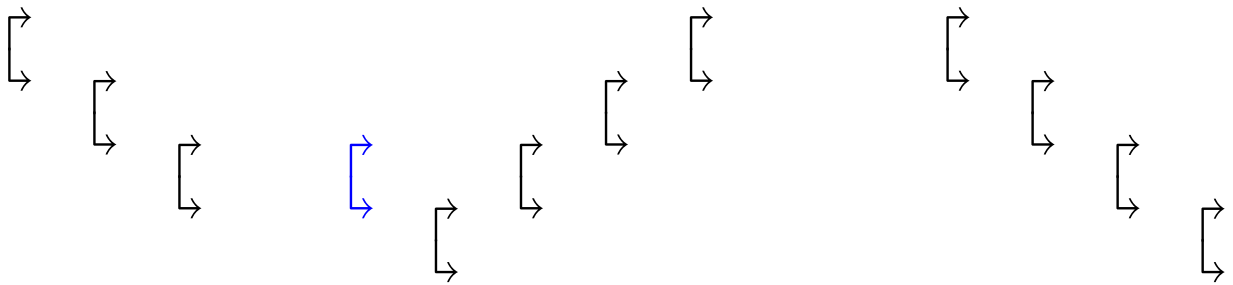

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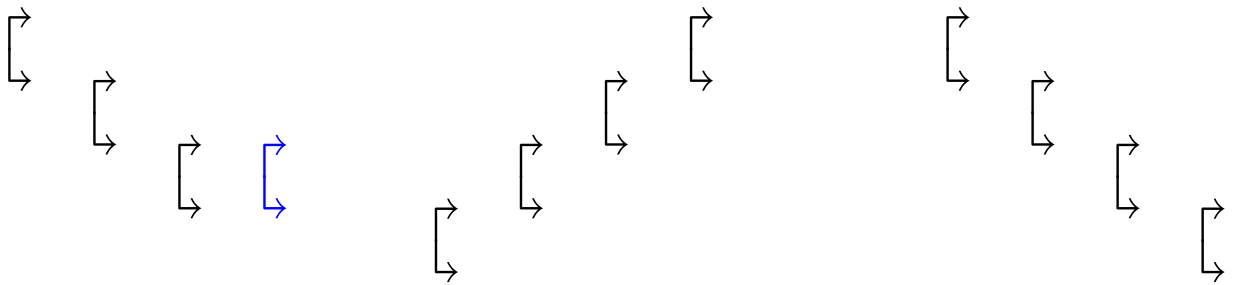

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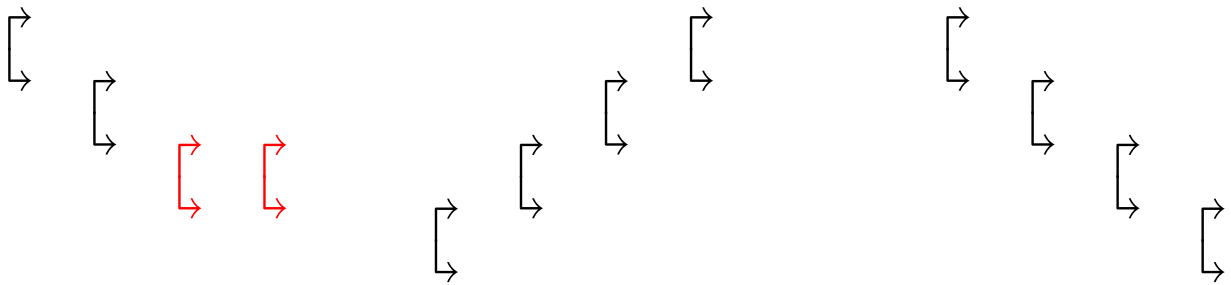

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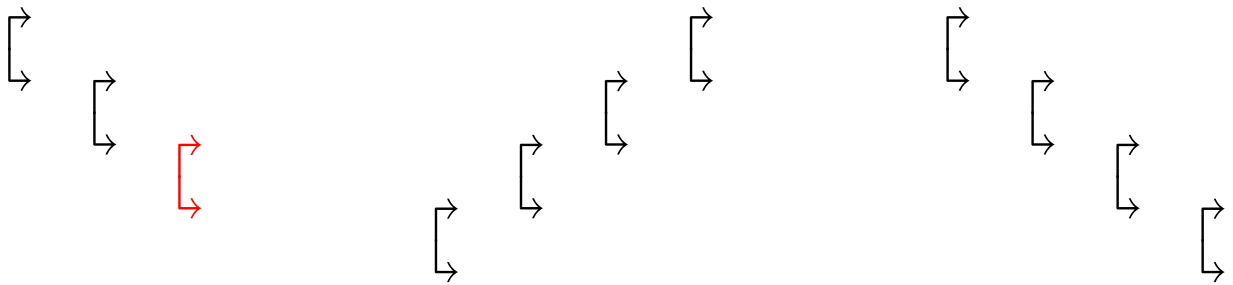

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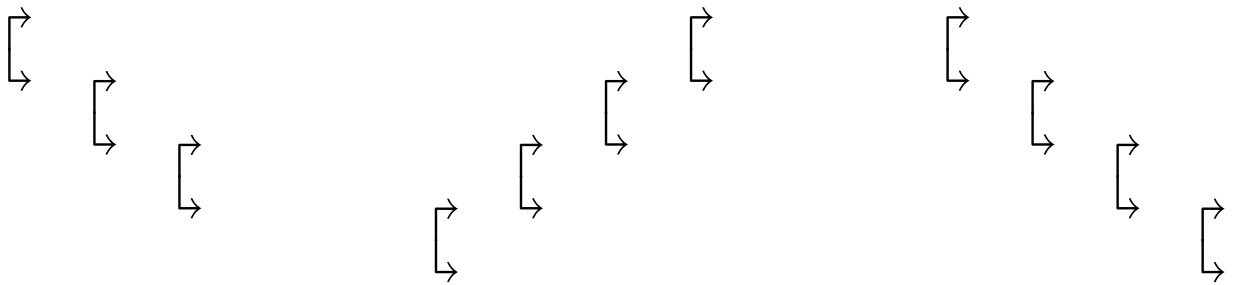

Done!

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- Cost: $3n$ turnovers/iteration, so $O(n)$ flops/iteration

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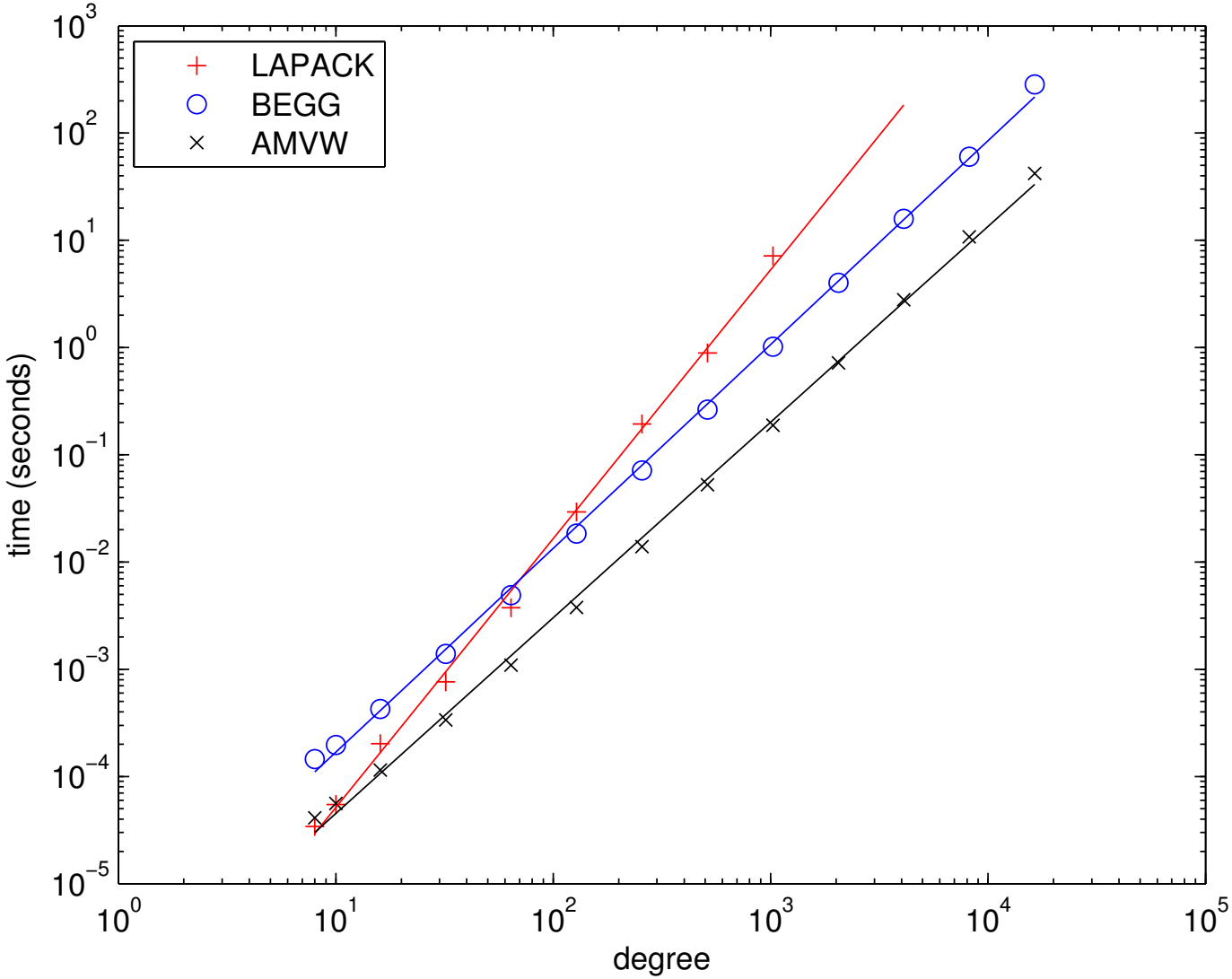
- Double-shift iteration is similar.
- (Chase two core transformations instead of one.)

Performance



Performance

At degree 1000

method	time
LAPACK	7.2
BEGG	1.2
AMVW	0.2

See our paper for . . .

- Paper in SIAM J. Matrix Anal. Appl. has

See our paper for . . .

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- . . . more timings,

See our paper for . . .

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- . . . more timings,
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- . . . more timings,
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Thank you for your attention.