Powers and More Powers

David S. Watkins

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University of Calgary, March 17, 2017

A bit of my history

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- Growing up in California . . .

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- I want to be

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- Growing up in California . . .
- I want to be a baseball star,

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- Growing up in California ...
- I want to be a baseball star, a teacher,

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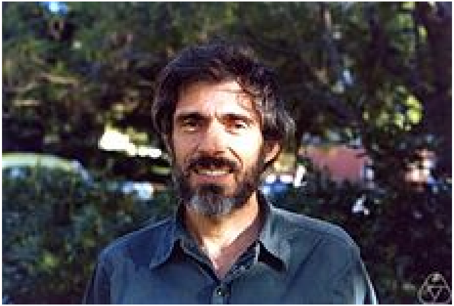

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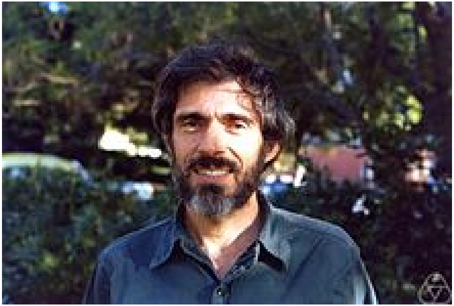

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- I ask Chandler Davis for advice.

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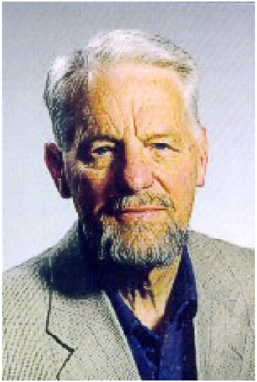

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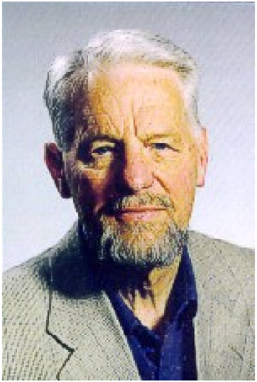

- So I came to Calgary.

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- I think about the job market.

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- Ph.D. University of Calgary, 1974.
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- . . . and I am on my way!

My trajectory

- Finite Elements (PDE)

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- Eigenvalues

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- Computing eigenvalues of *matrices*

My trajectory

- Finite Elements (PDE)
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- Computing eigenvalues of *matrices*
- I get stuck.

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- Compute some eigenvalues, eigenvectors, invariant subspaces, ...

Solving Eigenvalue Problems

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- What we don't do:

Solving Eigenvalue Problems

- $A \in \mathbb{C}^{n \times n}$
- Compute some eigenvalues, eigenvectors, invariant subspaces, ...
- What we don't do:
- We don't ever form the characteristic polynomial.

- So what do we do?

Solving Eigenvalue Problems

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- $v, Av, A^2v, A^3v, \ldots$
- This method is forgetful.

Solving Eigenvalue Problems

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Solving Eigenvalue Problems

- Arnoldi Process (Variant of Gram-Schmidt)
- kth step (We already have o.n. $q_1, \ldots, q_k$)
- $\hat{q}_{k+1} = Aq_k - \sum_{j=1}^{k} q_j h_{jk}$
- Normalize $\hat{q}_{k+1}$ to get q_{k+1}.

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- Some of these approximate peripheral eigenvalues of A.

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- We can run the Lanczos process,
- potentially forever.

Stieltjes Procedure (1884)

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- How to choose sample points and weights?
- One answer: maximize degree.

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- Now what?

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- So . . . symmetric Lanczos = Stieltjes.
- This is a consequence of the spectral theorem for self-adjoint operators.

Spectral Theorem

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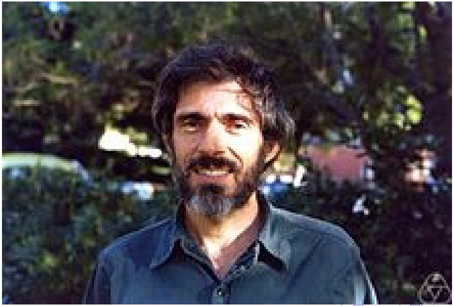

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- isomorphism: $p(A)v \to p(x)$ (Find the right μ.)
- $p(A)v \to A\,p(A)v$ maps to $p(x) \to x\,p(x)$
- Thus the action of A maps to multiplication by x.

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- This is exactly the Stieltjes procedure!
- Conclusion: The symmetric Lanczos process and the Stieltjes procedure are exactly the same thing.

My role in this

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It's been fun

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- What's next for this

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- What's next for this baseball star,

It's been fun

- What's next for this baseball star, teacher,

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- What's next for this baseball star, teacher, numerical analyst?

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- What's next for this baseball star, teacher, numerical analyst?
- Retirement in May.

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- What's next for this baseball star, teacher, numerical analyst?
- Retirement in May. Then what?

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- What’s next for this baseball star, teacher, numerical analyst?
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- Perhaps I should move back to Canada!

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Thank you for your attention.