SIMULTANEOUS NONVANISHING OF $GL(2) \times GL(2)$ AND $GL(2)$ $L$-FUNCTIONS

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Abstract. Let $f$ be a fixed holomorphic Hecke cusp form for $SL(2, \mathbb{Z})$. We prove that, for each $K$ large enough, there exists a holomorphic Hecke cusp form $g$ of weight $k$ with $K \leq k \leq 2K$ such that $L\left(\frac{1}{2}, g\right) L\left(\frac{1}{2}, f \times g\right) \neq 0$.

1. Introduction

The vanishing or nonvanishing of an automorphic $L$-function is an interesting problem in number theory. In many cases, it can encode a deep arithmetic property. For example, the Birch and Swinnerton-Dyer conjecture relates the order of vanishing of the Hasse-Weil $L$-function at the central point to the rank of an elliptic curve.

Let $\pi$ be a unitary cuspidal automorphic representation of $GL(n)$ and let $s_0 \in \mathbb{C}$. The nonvanishing of $L(s_0, \pi)$ twists by a Dirichlet character $\chi$ has been studied by Rohrlich [Ro], Barthel-Ramakrishnan [BR], Luo [Lu], Luo-Rudnick-Sarnak[LRS]. Especially, this problem is related to the Ramanujan conjecture (see [LRS]). The simultaneous nonvanishing of twists of automorphic $L$-functions was studied by many authors ([MV], [RR], [Li], [Kh], [Xu]). In her paper [Li], Li proved a simultaneous nonvanishing result for $GL(3) \times GL(2)$ and $GL(2)$ $L$-functions at the central point. More precisely, let $F$ be a fixed $GL(3)$ Maass cusp form. She showed that there are infinitely many $GL(2)$ Maass-Hecke cusp forms $u_j$ such that $L\left(\frac{1}{2}, F \times u_j\right) L\left(\frac{1}{2}, u_j\right) \neq 0$. Later Khan [Kh] proved a similar result with replacing the twists by $GL(2)$ holomorphic forms of prime level. In this paper, we will establish a simultaneous nonvanishing result for $GL(2) \times GL(2)$ and $GL(2)$ $L$-functions (Theorem 1).

Let $H_k$ denote the set of holomorphic Hecke cusp forms $g$ of weight $k$ for $SL(2, \mathbb{Z})$, where $g(z)$ has the Fourier expansion

$$g(z) = \sum_{n=1}^{\infty} \lambda_g(n)n^{k-1}e^{2\pi inz},$$

with $\lambda_g(1) = 1$.

Our main theorem is the following:

**Theorem 1.** Let $f$ be a fixed holomorphic Hecke cusp form for $SL(2, \mathbb{Z})$. For each $K$ large enough, there exists $g \in H_k$ with $K \leq k \leq 2K$ such that

$$L\left(\frac{1}{2}, f \times g\right) L\left(\frac{1}{2}, g\right) \neq 0.$$
Theorem 2. Let $f$ be a fixed holomorphic Hecke cusp form for $SL(2, \mathbb{Z})$. Let $u(\xi) \in C^\infty_c(0, \infty)$. Then for $K$ large, we have

$$\sum_{k \geq 2, 4|k} u \left( \frac{k-1}{K} \right) \sum_{g \in H_k} \omega_g^{-1} L \left( \frac{1}{2}, f \times g \right) L \left( \frac{1}{2}, g \right)$$

$$= K (\log K) L(1, f) \int_0^\infty u(\xi) d\xi + K \int_0^\infty u(\xi) \left( \log \xi + \gamma - \log 4\pi \right) L \left( \frac{1}{2}, f \right) \frac{1}{2} (1, f) \right) d\xi + O(K^\varepsilon),$$

where $\gamma$ is the Euler’s constant and the implied constant depends on $f$ and $\varepsilon$.

Remarks.

(1) $L(1, f) \neq 0$ (see [JS]).

(2) We only consider $k \equiv 0 \pmod{4}$, since for $k \equiv 2 \pmod{4}$, the root number of $L(s, g)$ is $-1$ and hence $L(\frac{1}{2}, g) = 0$.

2. Preliminaries

2.1. $GL(2)$ holomorphic cusp forms. The following Petersson trace formula is well-known and can be found in Iwaniec’s book [Iw].

Proposition 1 (Petersson trace formula).

$$\sum_{g \in H_k} \omega_g^{-1} \lambda_g(m) \lambda_g(n) = \delta_{m,n} + 2\pi i^{-k} \sum_{c=1}^\infty \frac{S(m, n; c)}{c} J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right),$$

where $\omega_g = (4\pi)^{k-1} ||g||^2$, $\delta_{m,n}$ equals 1 if $m = n$ and 0 otherwise, $S(m, n; c)$ is the Kloosterman sum defined below, and $J_{k-1}$ is the $J$-Bessel function.

The Kloosterman sum is defined as

$$S(m, n; c) = \sum_{a \equiv 1 (mod c)} e \left( \frac{ma + n\bar{a}}{c} \right),$$

and A. Weil proved that

$$|S(m, n; c)| \leq (m, n, c)^{1/2} e^{\frac{1}{2}} \tau(c),$$

where $\tau(c)$ is the divisor function.

Lemma 1. For any $\varepsilon > 0$, we have

$$\sum_{n \leq x} |S(m, n; c)| \ll xc^{1/2+\varepsilon}$$

where the implied constant depends only on $\varepsilon$.

Proof. By Weil’s bound, we have

$$\sum_{n \leq x} |S(m, n; c)| \ll c^{1/2+\varepsilon} \sum_{n \leq x} (n, c)^{1/2}$$

$$\ll c^{1/2+\varepsilon} \sum_{d|c} \sum_{n \leq x} d^{1/2} \ll xc^{1/2+\varepsilon}. $$
For each \( g \in H_k \), the \( L \)-function associated to \( g \) is defined by

\[
L(s, g) = \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^s}.
\]

It is entire and satisfies the functional equation

\[
\Lambda(s, g) = i^k \Lambda(1 - s, g)
\]

where

\[
\Lambda(s, g) = \pi^{-s} \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+k+1}{2}\right) L(s, g).
\]

**Lemma 2** (Approximate functional equation). Let \( k \equiv 0 \pmod{4} \) and let \( G(u) = e^{u^2} \). We have

\[
L\left(\frac{1}{2}, g\right) = 2 \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^{1/2}} V(n, k),
\]

where

\[
V(y, k) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma_1\left(\frac{1}{2} + u, k\right)}{\gamma_1\left(\frac{1}{2}, k\right)} du,
\]

and

\[
\gamma_1(s, k) = \pi^{-s} \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+k+1}{2}\right)
\]

**Proof.** See [IK, Theorem 5.3].

**Lemma 3.** 1) The derivatives of \( V(y, k) \) with respect to \( y \) satisfy

\[
y^a \frac{\partial^a}{\partial y^a} V(y, k) \ll \left(1 + \frac{y}{k}\right)^{-A}.
\]

The implied constant depends only on \( a \) and \( A \).

2) If \( 1 \leq y \ll k^{1+\varepsilon} \), then as \( k \to \infty \)

\[
V(y, k) = V_1\left(\frac{y}{k}\right) + O\left(\frac{1}{k} V_1^*\left(\frac{y}{k}\right)\right)
\]

where

\[
V_1(x) = \frac{1}{2\pi i} \int_{(3)} x^{-u} G_0(u) \frac{du}{u}, \quad V_1^*(x) = \frac{1}{2\pi i} \int_{(3)} x^{-u} G_1(u) \frac{du}{u}
\]

and \( G_0(u), G_1(u) \) are holomorphic functions with exponential decay as \( |\Re u| \to \infty \).

**Proof.** 1) See [IK, Proposition 5.4].

2) By Stirling’s formula

\[
G(u) \gamma_1\left(\frac{1}{2} + u, k\right) = G(u) \left(\frac{k}{4\pi}\right)^u \left(1 + O\left(\frac{p(u)}{k}\right)\right)
\]

\[
= k^u G_0(u) + O\left(\frac{1}{k} k^u G_1(u)\right),
\]

where \( p(u) \) is a polynomial in \( u \), \( G_0(u) = G(u)(4\pi)^{-u} \) and \( G_1(u) = G(u)p(u)(4\pi)^{-u} \). The result follows by inserting this into (2.3).
2.2. Rankin-Selberg $L$-functions. For $f \in H_l$ and $g \in H_k$, the Rankin-Selberg $L$-function of $f$ and $g$ is defined by

$$L(s, f \times g) = \zeta(2s) \sum_{m=1}^{\infty} \frac{\lambda_f(m) \lambda_g(m)}{m^s} = \sum_{m=1}^{\infty} \frac{\lambda_{f \times g}(m)}{m^s}$$

where

$$\lambda_{f \times g}(m) = \sum_{ab^2 = m} \lambda_f(a) \lambda_g(a).$$

It is entire and satisfies the functional equation

$$\Lambda(s, f \times g) = \Lambda(1 - s, f \times g)$$

where

$$\Lambda(s, f \times g) = (2\pi)^{-2s} \Gamma\left(s + \frac{k + l}{2}ight) \Gamma\left(s + \frac{k - l}{2}\right) L(s, f \times g).$$

Lemma 4 (Approximate functional equation). Let $G(u) = e^{u^2}$. We have

$$L\left(\frac{1}{2}, f \times g\right) = 2 \sum_{m=1}^{\infty} \frac{\lambda_{f \times g}(m)}{m^{1/2}} U(m, k),$$

where

$$U(y, k) = \frac{1}{2\pi i} \int_{(3)} y^{-u} G(u) \frac{\gamma_2\left(\frac{1}{2} + u, k\right)}{\gamma_2\left(\frac{1}{2}, k\right)} \frac{du}{u}.$$ 

and

$$\gamma_2(s, k) = (2\pi)^{-2s} \Gamma\left(s + \frac{k + l}{2} - 1\right) \Gamma\left(s + \frac{k - l}{2}\right)$$

Proof. See [IK, Theorem 5.3].

Lemma 5. 1) The derivatives of $U(y, k)$ with respect to $y$ satisfy

$$y^a \frac{\partial^a}{\partial y^a} U(y, k) \ll \left(1 + \frac{y}{k^2}\right)^{-A}.$$ 

The implied constant depends only on $a$ and $A$.

2) If $1 \leq y \ll k^{2+\epsilon}$, then as $k \to \infty$

$$U(y, k) = U_1\left(\frac{y}{k^2}\right) + O\left(\frac{1}{k^3} U_1^*\left(\frac{y}{k^2}\right)\right)$$

where

$$U_1(x) = \frac{1}{2\pi i} \int_{(3)} x^{-u} G_2(u) \frac{du}{u}, \quad U_1^*(x) = \frac{1}{2\pi i} \int_{(3)} x^{-u} G_3(u) \frac{du}{u}$$

and $G_2(u)$, $G_3(u)$ are holomorphic functions with exponential decay as $|3u| \to \infty$.

Proof. 1) See [IK, Proposition 5.4].

2) By Stirling’s formula

$$G(u) \frac{\gamma_2\left(\frac{1}{2} + u, k\right)}{\gamma_2\left(\frac{1}{2}, k\right)} = G(u) \left(\frac{k^2}{16\pi^2}\right)^u \left(1 + O\left(\frac{p(u)}{k}\right)\right)$$

$$= k^{2u} G_2(u) + O\left(\frac{1}{k} k^{2u} G_3(u)\right).$$
where \( p(u) \) is a polynomial in \( u \), \( G_2(u) = G(u)(16\pi^2)^{-u} \) and \( G_3(u) = G(u)p(u)(16\pi^2)^{-u} \). The result follows by inserting this into (2.6).

\[ \square \]

3. Proof of Theorem 2

By the approximate functional equations (2.2), (2.5) and the Petersson trace formula (Proposition 1), we have

\[
\sum_{g \in H_k} \omega_g^{-1} L\left( \frac{1}{2}, f \times g \right) L\left( \frac{1}{2}, g \right) = 4 \sum_{g \in H_k} \omega_g^{-1} \sum_{m=1}^{\infty} \frac{\lambda_f \times g(m)}{m^{1/2}} U(m, k) \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^{1/2}} V(n, k)
\]

\[
= 4 \sum_{g \in H_k} \omega_g^{-1} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{\lambda_f(a) \lambda_g(a)}{(ab^2)^{1/2}} U(ab^2, k) \sum_{n=1}^{\infty} \frac{\lambda_g(n)}{n^{1/2}} V(n, k)
\]

\[
= 4 \sum_{n=1}^{\infty} \sum_{b=1}^{\infty} \frac{1}{n^{1/2}} V(n, k) \frac{\lambda_f(n)}{(nb^2)^{1/2}} U(nb^2, k) + 8\pi \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \frac{1}{n^{1/2}} V(n, k) \frac{\lambda_f(a)}{(ab^2)^{1/2}} U(ab^2, k) \sum_{c=1}^{\infty} \frac{S(n, a; c)}{c} J_{k-1} \left( \frac{4\pi \sqrt{n}}{c} \right)
\]

\[ := 4D_k + 8\pi E_k. \]

3.1. Diagonal terms. We first deal with the diagonal terms and these will contribute the main term. Let

\[
D := 4 \sum_{k \geq 2, \text{\[4\mid k]} \left( k-1 \right) D_k.
\]

Lemma 6. We have

\[ D_k = (\log k + \gamma - \log 4\pi)L(1, f) + \frac{1}{2} L'(1, f) + O(k^{-1}). \]

and

\[ D = K(\log K)L(1, f) \int_0^\infty u(\xi) d\xi +
\]

\[ K \int_0^\infty u(\xi) \left( (\log \xi + \gamma - \log 4\pi)L(1, f) + \frac{1}{2} L'(1, f) \right) d\xi + O(1). \]

Proof.

\[
D_k = \sum_{n=1}^{\infty} \sum_{b=1}^{\infty} \frac{1}{n^{1/2}} V(n, k) \frac{\lambda_f(n)}{(nb^2)^{1/2}} U(nb^2, k)
\]

\[
= \frac{1}{(2\pi i)^2} \int_{(3)} \int_{(3)} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1+u+w}} \sum_{b=1}^{\infty} \frac{1}{b^{1+2w}} \frac{\gamma_1(\frac{1}{2} + u, k) \gamma_2(\frac{1}{2} + w, k)}{\gamma_1(u, k) \gamma_2(w, k)} \frac{G(u)G(w)}{u \ w} \frac{du \ dw}{u \ w}
\]

\[
= \frac{1}{(2\pi i)^2} \int_{(3)} \int_{(3)} L(1 + u + w, f) \zeta(1 + 2w) \frac{\gamma_1(\frac{1}{2} + u, k) \gamma_2(\frac{1}{2} + w, k)}{\gamma_1(u, k) \gamma_2(w, k)} \frac{G(u)G(w)}{u \ w} \frac{du \ dw}{u \ w}.
\]
Moving the line of integration to \( \Re u = -2 \) and \( \Re w = -\frac{1}{2} \), we pick up a simple pole at \( u = 0 \) and a double pole at \( w = 0 \). By the Residue theorem, we have

\[
D_k = \left( \frac{1}{2} \frac{\gamma_2'(\frac{1}{2}, k)}{\gamma_2(\frac{1}{2}, k)} + \gamma \right) L(1, f) + \frac{1}{2} L'(1, f) +
\frac{1}{2\pi i} \int_{(-2)} \left( \left( \frac{1}{2} \frac{\gamma_2'(\frac{1}{2}, k)}{\gamma_2(\frac{1}{2}, k)} + \gamma \right) L(1 + u, f) + \frac{1}{2} L'(1 + u, f) \right) \frac{\gamma_1(\frac{1}{2} + u, k)}{\gamma_1(\frac{1}{2}, k)} G(u) \frac{du}{u}
\]

\[
+ \frac{1}{2\pi i} \int_{(-1/2)} L(1 + w, f) \zeta(1 + 2w) \frac{\gamma_2(\frac{1}{2} + w, k)}{\gamma_2(\frac{1}{2}, k)} G(w) \frac{dw}{w}
\]

\[
+ \frac{1}{(2\pi i)^2} \int_{(-2)} \int_{(-1/2)} L(1 + u + w, f) \zeta(1 + 2w) \frac{\gamma_1(\frac{1}{2} + u, k)}{\gamma_1(\frac{1}{2}, k)} \frac{\gamma_2(\frac{1}{2} + w, k)}{\gamma_2(\frac{1}{2}, k)} G(u) G(w) \frac{du}{u} \frac{dw}{w}.
\]

By Stirling’s formula, we have

\[
\frac{\gamma_2'(\frac{1}{2}, k)}{\gamma_2(\frac{1}{2}, k)} = 2 \log k - 4 \log 2 - 2 \log \pi + O(k^{-1}),
\]

\[
\frac{\gamma_1(\frac{1}{2} + u, k)}{\gamma_1(\frac{1}{2}, k)} \ll k^{-2}
\]

for \( \Re u = -2 \), and

\[
\frac{\gamma_2(\frac{1}{2} + w, k)}{\gamma_2(\frac{1}{2}, k)} \ll k^{-1}
\]

for \( \Re w = -\frac{1}{2} \). (3.1) follows from using above estimates. For (3.2), by Poisson summation formula, we have

\[
(3.3) \quad 4 \sum_{k \geq 2, 4 \mid k} u \left( \frac{k - 1}{K} \right) = K \int_0^\infty u(\xi) d\xi + O(K^{-A})
\]

for any \( A > 0 \). By writing \( \log k = \log(\frac{k-1}{K}) + O(k^{-1}) \) and using (3.3) for \( u(\xi) \log \xi \), we obtain (3.2). \( \square \)

### 3.2. Off-diagonal terms.
We will prove the off-diagonal terms only contribute the error terms. Let

\[
E := \sum_{4 \mid k} u \left( \frac{k - 1}{K} \right) E_k
\]

\[
= \sum_{n=1}^\infty \sum_{a=1}^\infty \sum_{b=1}^\infty \frac{1}{n^{1/2}} \frac{\lambda_f(a)}{L(1/2, (ab^2)^1/2)} \sum_{c=1}^\infty \frac{S(n, a; c)}{c} \sum_{4 \mid k} u \left( \frac{k - 1}{K} \right) V(n, k) U(ab^2, k) J_{k-1} \left( \frac{4\pi \sqrt{na}}{c} \right).
\]

**Lemma 7.** For any \( \varepsilon > 0 \), we have

\[
E \ll K^\varepsilon.
\]
Proof. By taking a smooth dyadic subdivision, it suffices to estimate

\[ E_{N,M} = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w \left( \frac{n}{N} \right) h \left( \frac{ab^2}{M} \right) \frac{1}{n^{1/2}} \frac{1}{(ab^2)^{1/2}} \sum_{c=1}^{\infty} S(n,a;c) \times \]

\[ \sum_{4|k} u \left( \frac{k-1}{K} \right) V(n,k)U(ab^2,k)J_{k-1} \left( \frac{4\pi\sqrt{na}}{c} \right) \]

for \( N \ll K^{1+\varepsilon} \) and \( M \ll K^{2+\varepsilon} \), where \( w(\xi) \) and \( h(\xi) \) are fixed smooth functions with support contained in \([1,2]\).

Let

\[ R_1 = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w \left( \frac{n}{N} \right) h \left( \frac{ab^2}{M} \right) \frac{1}{n^{1/2}} \frac{1}{(ab^2)^{1/2}} \sum_{c \geq 16\pi\sqrt{Na}} S(n,a;c) \times \]

\[ \sum_{4|k} u \left( \frac{k-1}{K} \right) V(n,k)U(ab^2,k)J_{k-1} \left( \frac{4\pi\sqrt{na}}{c} \right) \]

and

\[ R_2 = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w \left( \frac{n}{N} \right) h \left( \frac{ab^2}{M} \right) \frac{1}{n^{1/2}} \frac{1}{(ab^2)^{1/2}} \sum_{c < 16\pi\sqrt{Na}} S(n,a;c) \times \]

\[ \sum_{4|k} u \left( \frac{k-1}{K} \right) V(n,k)U(ab^2,k)J_{k-1} \left( \frac{4\pi\sqrt{na}}{c} \right) . \]

Then \( E_{N,M} = R_1 + R_2 \). We will show \( R_1 \) is negligible and \( R_2 \ll K^{\varepsilon} \) in the next two sections. \( \square \)

3.3. Estimate of \( R_1 \). For \( c \geq 16\pi\sqrt{Na} \) and \( N \leq n \leq 2N \), we have

\[ x = \frac{4\pi\sqrt{na}}{c} \leq \frac{1}{4} \left( \frac{n}{N} \right)^{1/2} < e^{-1}. \]

Using Weil’s bound for Kloosterman sums, the bound \( J_{k-1}(x) \ll x^{k-1} \) for \( 0 < x < 1 \) and

\[ \sum_{n \leq x} |\lambda_f(n)| \ll x \]

we have

\[ R_1 \ll \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \left| w \left( \frac{n}{N} \right) \left| h \left( \frac{ab^2}{M} \right) \right| \frac{1}{n^{1/2}} \frac{|\lambda_f(a)|}{(ab^2)^{1/2}} \sum_{c \geq 16\pi\sqrt{Na}} c^{-1/2+\varepsilon} (n,a,c)^{1/2} \times \]

\[ \sum_{4|k} |u \left( \frac{k-1}{K} \right)| \left( \frac{4\pi\sqrt{na}}{c} \right)^{k-1} \]

\[ \ll \sum_{N \leq n \leq 2N} \frac{1}{n^{1/2}} \sum_{M \leq ab^2 \leq 2M} \frac{|\lambda_f(a)|}{(ab^2)^{1/2}} \sum_{c \geq 16\pi\sqrt{Na}} c^{-\varepsilon} \left( \frac{4\pi\sqrt{na}}{c} \right)^{2} \sum_{4|k, k \sim K} e^{-(k-3)} \]

\[ \ll K^{-A} \]

for any \( A > 0 \). \( \square \)
4. Estimate of $R_2$

Lemma 8. Fix a function $g(\xi) \in C_c^\infty(0, \infty)$. We have

$$\sum_{4 | k} g(k - 1)J_{k-1}(x) = -\frac{1}{2} F_1(x) - \frac{i}{2} F_2(x)$$

where

$$F_1(x) = \int_{-\infty}^{\infty} \hat{g}(t) \sin(x \cos(2\pi t)) \, dt,$$

$$F_2(x) = \int_{-\infty}^{\infty} \hat{g}(t) \sin(x \sin(2\pi t)) \, dt$$

and

$$\hat{g}(t) = \int_{-\infty}^{\infty} g(\xi) e(t\xi) \, d\xi$$

is the Fourier transform of $g(\xi)$.

Proof. See [Iw, P.85-87].

Let $g(\xi) = u(\xi/K)V(n, \xi + 1)U(ab^2, \xi + 1)$. By Lemma 8, $R_2 = -\frac{1}{2} H_1 - \frac{i}{2} H_2$ where

$$H_1 = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w\left(\frac{n}{N}\right) h\left(\frac{ab^2}{M}\right) \frac{1}{n^{1/2} (ab^2)^{1/2}} \sum_{c < 16\pi \sqrt{N \alpha}} \frac{S(n, a; e)}{c} F_1\left(4\pi \sqrt{na} \frac{c}{e}\right)$$

and

$$H_2 = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w\left(\frac{n}{N}\right) h\left(\frac{ab^2}{M}\right) \frac{1}{n^{1/2} (ab^2)^{1/2}} \sum_{c < 16\pi \sqrt{N \alpha}} \frac{S(n, a; e)}{c} F_2\left(4\pi \sqrt{na} \frac{c}{e}\right).$$

Let

$$g_0(\xi) = u(\xi)V(n, K \xi + 1)U(ab^2, K \xi + 1).$$

An easy change of variables gives $\hat{g}(t) = K \hat{g}_0(Kt)$ and

$$F_1(x) = \int_{-\infty}^{\infty} \hat{g}_0(t) \sin\left(x \cos\left(\frac{2\pi t}{K}\right)\right) \, dt,$$

$$F_2(x) = \int_{-\infty}^{\infty} \hat{g}_0(t) \sin\left(x \sin\left(\frac{2\pi t}{K}\right)\right) \, dt.$$

4.1. Estimate of $H_1$.

Lemma 9. For any $\varepsilon > 0$, we have

$$H_1 \ll K^{-1+\varepsilon}.$$

Proof. Let

$$W_1(x) = \int_{-\infty}^{\infty} \hat{g}_0(t) e\left(\frac{x}{2\pi} \left(1 - \cos\frac{2\pi t}{K}\right)\right) \, dt,$$

then

$$F_1(x) = \frac{-e(-\frac{x}{2\pi})W_1(x) + e(\frac{x}{2\pi})W_1(-x)}{2i}.$$
Since the contribution to $W_1(x)$ from $|t| \geq K^\epsilon$ is negligible, we only need to consider $|t| \leq K^\epsilon$.

Now we expand $\cos(\frac{2\pi t}{K})$ into Taylor series to get

$$W_1(x) = \int_{-\infty}^{\infty} \hat{g}_0(t)e\left(\frac{\pi xt^2}{K^2}\right) dt + O\left(\int_{-\infty}^{\infty} \left|\hat{g}_0(t)\right| \frac{|x|t^4}{K^4} dt\right)$$

Recall that $\hat{g}_0^{(a)}(t) = (-2\pi it)^a \hat{g}_0(t)$. Hence

$$W_1(x) = \int_{-\infty}^{\infty} \hat{g}_0(t)e\left(\frac{\pi xt^2}{K^2}\right) dt + O\left(\frac{|x|}{K^4}\right).$$

Using (2.1) and (3.4), the $O$-term in (4.1) contributes at most

$$\sum_{N \leq n \leq 2N} \sum_{M \leq ab^2 \leq 2M} \frac{1}{n^{1/2}} \frac{|\lambda_f(a)|}{(ab^2)^{1/2}} \sum_{c < 16\pi \sqrt{nA}} \frac{|S(n, a; c)|}{c} \frac{\sqrt{nA}}{c} K^{-4} \leq K^{-4} \sum_{M \leq ab^2 \leq 2M} \frac{|\lambda_f(a)|}{b} \sum_{c < 16\pi \sqrt{nA}} \frac{1}{c^2} \sum_{N \leq n \leq 2N} |S(n, a; c)| \leq N MK^{-4} \ll K^{-1+\epsilon}.$$  

By Parseval and [GR, 3.691 1],

$$\int_{-\infty}^{\infty} \hat{g}_0(t)e\left(\frac{\pi xt^2}{K^2}\right) dt = \frac{1 + i}{2} \int_{0}^{\infty} g_0(t) \frac{1}{\sqrt{\pi x}} e\left(-\frac{t^2}{4\pi K^2}\right) dt \leq \left(\frac{K^2}{|x|}\right)^{-A},$$

for any $A > 0$ (by using integration by parts).

Note that for $|x| = \frac{4\pi \sqrt{M}}{e} \ll N^{1/2}M^{1/2}$,

$$K^2 |x| \gg K^2 N^{-1/2} M^{-1/2} \gg K^{1/2-\epsilon}.$$ 

Hence the contribution from the integral in (4.1) to $H_1$ is negligible. \hfill \Box

4.2. Estimate of $H_2$. To estimate $H_2$, we need the following Voronoi formula on $GL(2)$ ([DI]).

**Proposition 2.** Let $F(\xi)$ be a smooth and compactly supported function on $\mathbb{R}^+$. For any integers $c \geq 1$ and $(c, d) = 1$ we have

$$\sum_{n} \lambda_f(n) e\left(\frac{dn}{c}\right) F(n) = \sum_{r} \lambda_f(r) e\left(-\frac{dr}{c}\right) \tilde{F}(r),$$

where $d \bar{d} \equiv 1 \pmod{c}$ and $\tilde{F}(r)$ is the Hankel-type transform

$$\tilde{F}(y) = 2\pi i c^{-1} \int_{0}^{\infty} F(\xi) J_{l-1}\left(\frac{4\pi \sqrt{\xi y}}{c}\right) d\xi,$$

and where $J_{\nu}(z)$ is the usual Bessel function.
Lemma 10. For any $\varepsilon > 0$, we have

$$H_2 \ll K^\varepsilon.$$  

Proof. Let 

$$W_2(x) = \int_{-\infty}^{\infty} \hat{g}_0(t) e \left( \frac{x}{2\pi} \sin \frac{2\pi t}{K} \right) dt$$

then 

$$F_2(x) = \frac{W_2(x) - W_2(-x)}{2i}.$$ 

Since the contribution to $W_2(x)$ from $|t| \geq K^\varepsilon$ is negligible, we only need to consider $|t| \leq K^\varepsilon$. Now expanding $\sin(\frac{2\pi x}{K})$ into Taylor series and using $\hat{g}_0(x) = g_0(-x)$, we get 

$$W_2(x) = \int_{-\infty}^{\infty} \hat{g}_0(t) e \left( \frac{xt}{K} \right) dt + O \left( \int_{-\infty}^{\infty} |\hat{g}_0(t)| \frac{|x|}{K^3} |t|^3 dt \right)$$

(4.2) 

$$= g_0 \left( - \frac{x}{K} \right) + O \left( \frac{|x|}{K^3} \right).$$ 

Using (2.1) and (3.4), the $O$-term in (4.2) contributes at most 

$$K^{-3} NM \ll K^\varepsilon.$$ 

Since $g_0(\xi)$ has compact support, $g_0 \left( - \frac{x}{K} \right)$ is nonzero only when $|x| \sim K$. For $|x| = \frac{4\pi \sqrt{na}}{c}$, $g_0 \left( - \frac{\xi}{K} \right)$ is nonzero only when $c \sim \frac{4\pi \sqrt{na}}{K} \sim \frac{\sqrt{NM}}{Kb}$ and the contribution to $H_2$ is 

$$\sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w \left( \frac{n}{N} \right) h \left( \frac{ab^2}{M} \right) \frac{\lambda_f(a)}{n^{1/2} (ab^2)^{1/2}} \sum_{c \sim \frac{\sqrt{NM}}{Kb}} S(n, a; c) g_0 \left( \frac{4\pi \sqrt{na}}{Kc} \right).$$ 

Note that 

$$g_0 \left( \frac{4\pi \sqrt{na}}{Kc} \right) = u \left( \frac{4\pi \sqrt{na}}{Kc} \right) V \left( n, \frac{4\pi \sqrt{na}}{c} + 1 \right) U \left( ab^2, \frac{4\pi \sqrt{na}}{c} + 1 \right).$$ 

By Lemma 3 and Lemma 5, we only need to consider the contribution from 

$$u \left( \frac{4\pi \sqrt{na}}{Kc} \right) V_1 \left( \frac{n}{\frac{4\pi \sqrt{na}}{c} + 1} \right) U_1 \left( \frac{ab^2}{\left( \frac{4\pi \sqrt{na}}{c} + 1 \right)^2} \right)$$ 

in (4.3), since the $O$-term is smaller. Let 

$$\tilde{H}_2 = \sum_{n=1}^{\infty} \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} w \left( \frac{n}{N} \right) h \left( \frac{ab^2}{M} \right) \frac{\lambda_f(a)}{n^{1/2} (ab^2)^{1/2}} \times$$ 

$$\sum_{c \sim \frac{\sqrt{NM}}{Kb}} S(n, a; c) u \left( \frac{4\pi \sqrt{na}}{Kc} \right) V_1 \left( \frac{n}{\frac{4\pi \sqrt{na}}{c} + 1} \right) U_1 \left( \frac{ab^2}{\left( \frac{4\pi \sqrt{na}}{c} + 1 \right)^2} \right).$$ 

After opening the Kloosterman sum and applying Proposition 2 to sum over $a$ with 

$$F(\xi) = h \left( \frac{\xi b^2}{M} \right) u \left( \frac{4\pi \sqrt{na}}{Kc} \right) V_1 \left( \frac{n}{\frac{4\pi \sqrt{na}}{c} + 1} \right) U_1 \left( \frac{\xi b^2}{\left( \frac{4\pi \sqrt{na}}{c} + 1 \right)^2} \right) \xi^{-1/2},$$
we have
\[ \widetilde{H}_2 = \sum_{n=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c \sim \frac{N}{K} d \mod c} w\left(\frac{n}{N}\right) \frac{1}{n^{1/2}bc} e\left(\frac{nd}{c}\right) \sum_{r=1}^{\infty} \lambda_f(r) e\left(-\frac{rd}{c}\right) \tilde{F}(r). \]

Note that \( F(\xi) \) has support contained in \( [\frac{M}{b^2}, 2\frac{M}{b^2}] \) and
\[ F^{(i)}(\xi) \ll \left(\frac{M}{b^2}\right)^{-1/2-i}. \]

Using the recurrence formula
\[ \frac{d}{dz}(z^\nu J_\nu(z)) = z^\nu J_{\nu-1}(z), \]
and the bound \( J_\nu(z) \ll (1 + z)^{-1/2} \), we have
\[ \tilde{F}(y) \ll c^{-1} y^{-7/4} N^{7/4} M^{1/2} K^{7/2} \ll y^{-7/4} K^{5/2}. \]

by integration by parts three times and (4.3). By (4.4) and Deligne’s bound \( \lambda(r) \ll r^\varepsilon \), we have
\[ \widetilde{H}_2 \ll \sum_{N \leq n \leq 2N} \sum_{b=1}^{\infty} \sum_{c \sim \frac{N}{K} d \mod c} \frac{1}{n^{1/2}bc} \sum_{r=1}^{\infty} r^{-7/4+i} N^{3/4} K^{-5/2} \ll K^{-3/4+\varepsilon}. \]

\[ \square \]

5. A note on simultaneous nonvanishing of \( GL(3) \times GL(2) \) and \( GL(2) \)

\( L \)-functions

Let \( F \) be a fixed \( GL(3) \) Maass-Hecke cusp form. Li [Li] and Khan [Kh] proved that there are infinitely many \( GL(2) \) cusp forms \( g \) such that \( L(\frac{1}{2}, F \times g) L(\frac{1}{2}, g) \neq 0 \). Their proofs were to establish an asymptotic formula like Theorem 2 which is very delicate in their cases. However if we just want to prove that there exists one \( g \) such that \( L(\frac{1}{2}, F \times g) L(\frac{1}{2}, g) \neq 0 \) for the special case \( F = \text{sym}^2 f \), where \( f \) is a \( GL(2) \) cusp form, it is much easier. Actually, it is a consequence of Watson’s formula [Wa]. More precisely, we have the following result.

**Theorem 3.** Let \( f \) be a fixed holomorphic Hecke cusp form of weight \( k \) for \( SL(2, \mathbb{Z}) \). Then there exists \( g \in H_{2k} \) such that

\[ L(\frac{1}{2}, \text{sym}^2 f \times g)L(\frac{1}{2}, g) \neq 0. \]

**Proof.** Since \( f^2 \) is a holomorphic cusp of weight \( 2k \), we can express it as
\[ f^2 = \sum_{g \in H_{2k}} \left\langle f^2, \frac{g}{||g||} \right\rangle \frac{g}{||g||}. \]

Using Watson’s formula [Wa, Theorem 3],
\[ \frac{|\left\langle f^2, g \right\rangle|^2}{\langle f, f \rangle^2 \langle g, g \rangle} = \frac{\Lambda(\frac{1}{2}, f \times f \times g)}{4 \Lambda(1, \text{sym}^2 f)^2 \Lambda(1, \text{sym}^2 g)} \]

\[ \square \]
(here $\Lambda(s, f \times f \times g)$, $\Lambda(s, \text{sym}^2 f)$ and $\Lambda(s, \text{sym}^2 g)$ are completed $L$-functions) and the factorization
\[ L(s, f \times f \times g) = L(s, \text{sym}^2 f \times g) L(s, g), \]
we deduce that
\[ 0 \neq \langle f^2, f^2 \rangle = \sum_{g \in H_{2k}} \frac{|\langle f^2, g \rangle|^2}{\langle g, g \rangle} = C_{k,f} \sum_{g \in H_{2k}} \frac{L(\frac{1}{2}, \text{sym}^2 f \times g) L(\frac{1}{2}, g)}{L(1, \text{sym}^2 g)} \tag{5.3} \]
where $C_{k,f}$ is a nonzero constant which depends on $k$ and $f$. The theorem follows from (5.3). □

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References


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