1. (a) State Duhamel's principle for the heat equation with B.C. $u_x(0,t)=0,\ u_x(L,t)=0$.
(b) Solve

$$\begin{aligned}&\text{D.E. } u_t - k u_{xx} = t\cos x, \quad 0\le x\le \pi, \quad t\ge 0\\ &\text{B.C. } u_x(0,t)=0, \quad u_x(\pi,t)=0\\ &\text{I.C. } u(x,0)=0.\end{aligned}$$

(a). Duhamel's principle: there are two solution $u(x,t)$ and $v(x,t;s)$ satisfy following respectively:

D.E	$u_t - k u_{xx} = h(x,t)$ $\quad 0\le x\le L,\ t\ge 0$	DE	$V_t - k V_{xx} = 0$ $\quad 0\le x\le L\ \ t\ge s$
B.C	$u_x(0,t) = u_x(L,t) = 0$	B.C	$V_x(0,t;s) = V_x(L,t;s) = 0$
I.C	$u(x,0) = 0$	I.C	$V(x,s;s) = h(x,s)$

If $u(x,t)$ and $V(x,t;s)$ are the solution of PDE above respectively, they have the relationship:

$$u(x,t) = \int_0^t V(x,t;s)\,ds.$$

(b). step 1. find out $\tilde{v}(x,t;s)$ which satisfies following.

D.E $\quad \tilde{V}_t - k\tilde{V}_{xx} = 0 \qquad 0\le x\le \pi,\quad t\ge 0$

B.C $\quad \tilde{V}_x(0,t;s) = \tilde{V}_x(\pi,t;s) = 0$

I.C $\quad \tilde{V}(x,0;s) = s\cos x$

solve $\tilde{v}(x,t;s)$ with B.C. $\qquad \tilde{V}(x,t;s) = s\,e^{-kt}\cos x$.

Step 2. use Duhamel's principle:

$$u(x,t) = \int_0^t V(x,t;s)\,ds = \int_0^t \tilde{V}(x,t-s;s)\,ds = \int_0^t s\,e^{-k(t-s)}\cos x\ ds$$

$$= e^{-kt}\cdot\cos x\int_0^t s\,e^{ks}\,ds$$

$$= e^{-kt}\cos x\cdot\left[\frac{1}{k}\cdot t e^{kt} - \frac{1}{k^2}(e^{kt}-1)\right]$$

$$= \left[\frac{t}{k} + \frac{e^{-kt}}{k^2} - \frac{1}{k^2}\right]\cdot\cos x \qquad (0\le x\le\pi,\quad t\ge 0)$$

2. Let $k > 0$. Consider the following problem:

$$\text{D.E. } u_t = ku_{xx}, \quad 0 \le x \le \pi, \quad t \ge 0$$
$$\text{B.C. } u(0,t) = 0, \quad u(\pi,t) = 0$$
$$\text{I.C. } u(x,0) = 5\sin(3x) - 3\sin(5x).$$

(a) Give a proof that there is at most one solution $u(x,t)$ to this problem.

(b) Show that the solution satisfies $-8 \le u(x,t) \le 4\sqrt{2}$ for all $0 \le x \le \pi,\ t \ge 0$.

(a) Assume there are two solutions $U_1(x,t)$ and $U_2(x,t)$

Consider $V(x,t) = U_1(x,t) - U_2(x,t)$, so we have following:

DE $V_t - kV_{xx} = U_{1t} - kU_{1xx} - U_{2t} + kU_{2xx} = 0$

B.C $V(0,t) = V(\pi,t) = 0$

I.C $V(x,0) = U_1(x,0) - U_2(x,0) = 0$

Now. consider $F(t) = \int_0^\pi (V(x,t))^2 dx$.

We have i) $F(t) \geq 0$ $\quad (\because (V(x,t))^2 \geq 0)$

ii) $F(0) = 0$ $\quad (\because F(0) = \int_0^0 (V(x,t))^2 dx = 0)$

iii) $F'(t) = \int_0^\pi 2V(x,t)\cdot V_t(x,t)\,dx = \int_0^\pi 2kV(x,t)\cdot V_{xx}\,dx = 2k\int_0^\pi V(x,t)\cdot dV_x(x,t)$

$= 2k\cdot V(x,t)\cdot V_x(x,t)\Big|_0^\pi - 2k\int_0^\pi (V_x(x,t))^2 dx$

$\because V(0,t) = V(\pi,t) = 0$

$\therefore F'(t) = -2k\int_0^\pi (V_x(x,t))^2 dt \le 0$ $\quad (\because (V_x(x,t))^2 \geq 0)$

combine i) ii) & iii). we have $F(t) \equiv 0$

$\therefore V(x,t) = 0$ $\quad \therefore U_1(x,t) = U_2(x,t)$ $\quad \therefore$ there at most one solution.

(b) Using Max/Min Principle.

$U(x,0) = f(x) = 5\sin 3x - 3\sin 5x$

$f'(x) = 15\cos 3x - 15\cos 5x$

$= -30\sin 4x \sin(-x)$

$= 30\sin 4x \sin x$

there are 5 points:

$0\ \frac{1}{4}\pi,\ \frac{1}{2}\pi\ \frac{3}{4}\pi,\ \pi$

i) $x = 0$ $\quad f(x) = 0$

ii) $x = \frac{1}{4}\pi$ $\quad f(x) = 5\cdot\frac{\sqrt{2}}{2} + 3\cdot\frac{\sqrt{2}}{2} = 4\sqrt{2}$

iii) $x = \frac{1}{2}\pi$ $\quad f(x) = -5 - 3 = -8$

IV) $x = \frac{3}{4}\pi$ $\quad f(x) = 5\cdot\frac{\sqrt{2}}{2} + 3\frac{\sqrt{2}}{2} = 4\sqrt{2}$

V) $x = \pi$ $\quad f(x) = 0$

$\therefore -8 \le f(x) \le 4\sqrt{2}$, $\quad 0 \le x \le \pi$

According to Max/Min principle.

$\min\{0, \min_{0\le x\le\pi}\{f(x)\}\} \le U(x,t) \le \max\{0, \max_{0\le x\le\pi}\{f(x)\}\}$

$\therefore -8 \le U(x,t) \le 4\sqrt{2}$ for all $x \in [0,\pi]$, $t \geq 0$

3. Find the parametric form of the solutions of the following PDE.

$$xu_x + 2yu_y = 0, \quad u(s, e^{-s}) = \sin s, \; s > 0.$$

solve. $x = X(s,t)$

$y = Y(s,t)$

$\therefore \begin{cases} X' = X \\ Y' = 2Y \end{cases}$

$\begin{cases} X(s,t) = C_1(s)\, e^{t} \\ Y(s,t) = C_2(s)\, e^{2t} \end{cases}$

with I.C.

$\begin{cases} X(s,0) = s \\ Y(s,0) = e^{-s} \end{cases}$

$\begin{cases} C_1(s) = s \\ C_2(s) = e^{-s} \end{cases}$

$\therefore \begin{cases} X(s,t) = s \cdot e^{t} \\ Y(s,t) = e^{2t-s} \end{cases}$ ✓

then

$U(s,t) = u(X(s,t), Y(s,t))$

$\therefore \; U_t(s,t) = 0$

$\therefore \; U(s,t) = g(s)$

with I.c.

$U(s,0) = g(s) = \sin s.$

$\therefore \; U(s,t) = \sin s$ ✓

$\therefore$ the parametric form solution is

$\begin{cases} X(s,t) = s \cdot e^{t} \\ Y(s,t) = e^{2t-s} \\ U(s,t) = \sin s \end{cases}, \quad s > 0$

✓

4. Show that the following PDE with side condition has no solution.

$$u_x + 3u_y - u = 1, \quad u(x, 3x) = 2.$$

solve. $a=1 \quad b=3$.

$\therefore \begin{cases} w = bx - ay = 3x - y \\ z = y \end{cases} \qquad \begin{cases} x = \frac{1}{b}(w + az) = \frac{1}{3}(w+z) \\ y = z \end{cases}$

the PDE become:

$3 \cdot V_z - V = 1$

$V_z - \frac{1}{3}V = \frac{1}{3}$

$m(z) = e^{-\frac{1}{3}z}$

$\therefore (e^{-\frac{1}{3}z} V)' = \frac{1}{3} e^{-\frac{1}{3}z}$

$e^{-\frac{1}{3}z} \cdot V = -e^{-\frac{1}{3}z} + g(w)$

$\therefore V = -1 + e^{\frac{1}{3}z} \cdot g(w)$

$u(x,y) = e^{\frac{1}{3}y} \cdot g(3x-y) - 1$ ✓

to determine $g(3x-y)$. we need to use B.C.

$u(x.3x) = 2$.

$u(x.3x) = e^x \cdot g(0) - 1 = 2$

$\therefore g(0) = 3 \cdot e^{-x}$

The left side is a constant, but the right side is a function of x, cannot find out $g(x)$ $\therefore$ There is no solution with this side condition

5. Find the all eginvalues and the corrsponding eigenfunctions of the Sturm-Liouville problem:

$$\text{D.E. } y'' + \lambda y = 0, \quad 0 \le x \le L$$
$$\text{B.C. } y'(0) = 0, \quad y(L) = 0.$$

char. eqn. $t^2 + \lambda = 0$

case 1) $\lambda > 0$ $\quad t = \pm i\sqrt{\lambda}$ $\quad y = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$ $\quad y' = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

case 2) $\lambda = 0$ $\quad t = 0$ $\quad y = C_1 x + C_2$ $\quad y' = C_1$

case 3) $\lambda < 0$ $\quad t = \pm\sqrt{-\lambda}$ $\quad y = C_1 e^{\sqrt{-\lambda}\cdot x} + C_2 e^{-\sqrt{-\lambda}\cdot x}$ $\quad y' = C_1\cdot\sqrt{-\lambda}\cdot e^{\sqrt{-\lambda}\cdot x} - C_2\sqrt{-\lambda}\cdot e^{-\sqrt{-\lambda}\cdot x}$

For case (1) $\quad y'(0) = 0 \quad y'(0) = C_2\cdot\sqrt{\lambda} = 0 \quad C_2 = 0.$

$y(L) = 0 \quad y(L) = C_1 \cos\sqrt{\lambda}L = 0$ to make $C_1 \neq 0$ $\quad \cos\sqrt{\lambda}L = 0$

$\sqrt{\lambda}L = (n+\frac{1}{2})\pi \quad \therefore \lambda = (n+\frac{1}{2})^2\frac{\pi^2}{L^2} \quad n = 0, 1, 2, 3 \cdots$

corresponding eigen function $y(x) = C_1 \cdot \cos\left((n+\frac{1}{2})\frac{\pi}{L}\cdot x\right)$ $\quad n = 0, 1, 2, 3 \cdots$

For case 2) $\quad y'(0) = 0 \quad C_1 = 0$

$y(L) = C_2 = 0$

there is no non-zero solution. $\therefore$ 0 is NOT egeinvalue.

For case 3) $\quad y'(0) = 0 \quad y'(0) = C_1\cdot\sqrt{-\lambda} - C_2\sqrt{-\lambda} = 0 \quad \therefore C_1 = C_2 \quad (\because \sqrt{-\lambda} \neq 0)$

$y(L) = 0 \quad y(L) = C_1\left(e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L}\right) = 0 \quad \therefore C_1 = 0$

$\therefore C_1 = C_2 = 0$

there is no negtive eigeinvalue.

Conclusion:

$\lambda_n = (n+\frac{1}{2})^2\frac{\pi^2}{L^2} \quad n = 0, 1, 2, 3 \cdots$ are the eigenvalue.

Corresponding eigenfunctions are

$y_n(x) = C_n \cdot \cos\left((n+\frac{1}{2})\frac{\pi x}{L}\right) \quad n = 0, 1, 2, 3 \cdots$ $\quad C_n$ are constant.