Numerical Evaluation of Singular Multivariate Normal Distributions

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Abstract

We present an efficient and accurate method to evaluate multivariate normal probabilities with arbitrary singular correlation matrices. The new method is applied to the construction of simultaneous confidence intervals and simultaneous all pairwise confidence intervals for multinomial proportions when the sample size is sufficiently large.

Key Words: multivariate normal, singular distribution, numerical integration, statistical computation.

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1 Introduction

Let $X_1, \ldots, X_m$ $(m \geq 2)$ be the standardized m-variate normal random variates with a correlation matrix $\{\rho_{jk}^{(m)}\}$. Consider the probability

$$\Pr\left[\bigcap_{j=1}^{m}\left(X_j \leq b_j\right);\ \{\rho_{jk}^{(m)} = \alpha_{jk}\}\right] \tag{1}$$

where $b_1, \ldots, b_m \in \Re$ and where $\{\rho_{jk}^{(m)} = \alpha_{jk}\}$ denotes the correlation matrix $\{\rho_{jk}^{(m)}\}$ with entry α_{jk} in the j-th row and k-th column for $j \neq k$ and entry 1 for $j = k$, where $1 \leq j, k \leq m$. The methods for evaluating the probability in (1) with various non-singular correlation structures have been extensively studied by Dunnett and Sobel (1955), Steck and Owen (1962), Schervish (1984, 1985), Nelson (1991), Dunnett (1989, 1993), Drezner (1992), Genz (1992), and Hajivassiliou, McFadden and Rudd (1996). For example, in order to evaluate an l-variate normal probability with a non-singular negative product structure ($\{\rho_{jk}^{(l)} = -\alpha_j\alpha_k\}$ where $l < m$ and $\sum_{j=1}^{m} \alpha_j^2/(1+\alpha_j^2) = 1$), Nelson (1991) proved that for any $2 \leq l < m$

$$\Pr\left[\bigcap_{j=1}^{l}\left(X_j \leq b_j\right);\ \{\rho_{jk}^{(l)} = -\alpha_j\alpha_k\}\right] = \int_{-\infty}^{\infty} \prod_{j=1}^{l}\left[\Phi\left(\frac{b_j - i\alpha_j z}{\sqrt{1+\alpha_j^2}}\right)\right]\phi(z)dz \tag{2}$$

where ϕ is the standard normal density function and Φ is the standard normal distribution function extended to complex domain and defined by

$$\Phi(x+iy) = e^{y^2/2}\int_{-\infty}^{x} e^{-isy}\phi(s)ds$$

where $i^2 = -1$. Kwong (1995) showed that the result in (2) is not valid for the singular correlation structure, when $l = m$. After modifying (2) for $l = m$, Kwong (1995) proved a new theorem that provides a method for evaluating one-sided multivariate normal probabilities with such singular correlation structure. However, the result cannot be extended to evaluate the two-sided probabilities in the form

$$\Pr\left[\bigcap_{j=1}^{m}\left(|X_j| \leq b_j\right);\ \{\rho_{jk}^{(m)} = -\alpha_j\alpha_k\}\right] \tag{3}$$

where $b_j > 0$ for $j = 1, \cdots, m$. Recently, Kwong and Iglewicz (1996) derived a new approach for evaluating (3) for $m = 3$, and with the additional restriction that $\alpha_1 = \alpha_2 = \cdots = \alpha_m = -1/(m-1)$ for $m \geq 4$. Kwong (1998) provided another approach to evaluate the upper and lower bounds for (3) for any $m \geq 4$.

In this paper, a new approach for evaluating (3) with any arbitrary singular correlation structures is presented. Numerical and simulation studies are conducted to compare the new approach with the existing method and simulation results. Then, the new approach is applied to evaluate the critical values for the construction of simultaneous confidence intervals and simultaneous all pairwise confidence intervals for multinomial proportions when the sample size is sufficiently large.

2 Singular Multivariate Normal Integral Evaluation

2.1 Negative Product Correlation Structure

Let $X_1, \ldots, X_m$ be the standardized m-variate normal variates with a singular negative product correlation structure, *i.e.* $\{\rho_{jk}^{(m)} = -\alpha_j\alpha_k\}$ with $\sum_{j=1}^{m} \alpha_j^2/(1+\alpha_j^2) = 1$. Denote the events $A_j = \{X_j : |X_j| \leq b_j\}$ for

$j = 1, \ldots, m$ and $J_r^m = \{(j_1, \ldots, j_r) : 1 \le j_1 < j_2 < \cdots < j_r \le m\}$ be a set in an r-dimensional space with all the j_l being integers. Kwong (1998) derived the following inequalities:

$$\begin{aligned}\sum_{r=1}^{m-2}(-1)^{r+1}\sum_{J_r^{m-1}}\Pr\left[\left(\bigcap_{l=1}^{r}A_{j_l}\right)\bigcap A_m\right]-\Pr[A_m] &\le \Pr\left[\bigcap_{j=1}^{m}A_j\right]\\ &\le 1-\sum_{r=1}^{m-1}(-1)^{r+1}\sum_{J_r^m}\Pr\left[\bigcap_{l=1}^{r}A_{j_l}\right]\end{aligned}$$

when m is an odd integer, and

$$\begin{aligned}\sum_{r=1}^{m-1}(-1)^{r+1}\sum_{J_r^m}\Pr\left[\bigcap_{l=1}^{r}A_{j_l}\right]-1 &\le \Pr\left[\bigcap_{j=1}^{m}A_j\right]\\ &\le \Pr[A_m]-\sum_{r=1}^{m-2}(-1)^{r+1}\sum_{J_r^{m-1}}\Pr\left[\left(\bigcap_{l=1}^{r}A_{j_l}\right)\bigcap A_m\right]\end{aligned}$$

when m is an even integer. Notice that the upper and lower bounds for the singular multivariate normal probability are all expressed in terms of the multivariate normal probabilities with non-singular negative product correlation structures. Therefore, the bounds can be numerically evaluated after the result in (2) is extended as follows:

$$\Pr\left[\bigcap_{j=1}^{l}A_j;\ \{\rho_{jk}^{(l)}=-\alpha_j\alpha_k\}\right]=\int_{-\infty}^{\infty}\prod_{j=1}^{l}\left[\Phi\left(\frac{b_j-i\alpha_j z}{\sqrt{1+\alpha_j^2}}\right)-\Phi\left(\frac{-b_j-i\alpha_j z}{\sqrt{1+\alpha_j^2}}\right)\right]\phi(z)dz.$$

when $2 \le l < m$ and $\Pr[A_j] = 2\Phi(b_j) - 1$ when $l = 1$.

Obviously, Kwong's inequalities can be applied to evaluate the upper and lower bounds of an m-variate normal distribution with any arbitrary singular correlation matrix of rank k ($k < m$) if the inequalities are used recursively until all the probabilities are expressed in terms of the nonsingular multivariate normal probabilities which are then computed by any of existing approaches. However, as $m - k$ increases, the difference between the bounds and the exact value, as well as the computational time, increases rapidly. Kwong's inequalities are therefore not an efficient and accurate approach to evaluate any singular multivariate normal probabilities when $m - k > 1$. A new approach is derived in the next section.

2.2 General Correlation Structure

We begin with the standard definition of the multivariate normal distribution when the variance-covariance matrix Σ is positive definite. Define

$$F(\mathbf{a},\mathbf{b},\Sigma)=\Pr\left[\bigcap_{j=1}^{m}\left(a_j\le X_j\le b_j\right);\ \Sigma\right]=\frac{(2\pi)^{-\frac{m}{2}}}{\sqrt{|\Sigma|}}\int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}e^{-\frac{1}{2}\mathbf{x}'\Sigma^{-1}\mathbf{x}}dx_m\cdots dx_1.$$

A key step in the development of the numerical methods described by Genz (1992) was to transform F using $\mathbf{x} = C\mathbf{y}$, where C is the Cholesky factor of Σ ($\Sigma = CC'$). After this transformation

$$F(\mathbf{a},\mathbf{b},\Sigma)=(2\pi)^{-\frac{m}{2}}\int_{\mathbf{a}\le C\mathbf{y}\le\mathbf{b}}e^{-\frac{1}{2}\mathbf{y}'\mathbf{y}}d\mathbf{y}.$$

We wish to generalise this approach for cases where Σ is positive semi-definite, and show that a similar definition of F can also be used in these cases. Let $\Sigma = UDU'$ be a singular value decomposition for Σ,

where U is an $m \times m$ orthogonal matrix and D is an $m \times m$ diagonal matrix with nonnegative diagonal entries $d_1, d_2, \ldots, d_m$. If k is the rank of Σ, and $k < m$, then $d_{k+1} = d_{k+2} = \ldots = d_m = 0$. Now define $\Sigma(\epsilon) = \Sigma + \epsilon I$, for $\epsilon > 0$. Then

$$\Sigma(\epsilon) = UDU' + \epsilon I = U(D + \epsilon I)U' = UD(\epsilon)U',$$

where $D(\epsilon) = D + \epsilon I$, so $\Sigma(\epsilon)$ is positive definite, and $F(\mathbf{a}, \mathbf{b}, \Sigma(\epsilon))$ is properly defined. Next let $V(\epsilon) = UD(\epsilon)^{\frac{1}{2}}$, so $\Sigma(\epsilon) = V(\epsilon)V'(\epsilon)$, and let $\mathbf{x} = V(\epsilon)\mathbf{v}$. Then

$$F(\mathbf{a}, \mathbf{b}, \Sigma(\epsilon)) = (2\pi)^{-\frac{m}{2}} \int_{\mathbf{a} \le V(\epsilon)\mathbf{v} \le \mathbf{b}} e^{-\frac{1}{2}\mathbf{v}'\mathbf{v}} d\mathbf{v}.$$

Taking the limit as ϵ approaches zero, we have

$$F(\mathbf{a}, \mathbf{b}, \Sigma) = (2\pi)^{-\frac{m}{2}} \int_{\mathbf{a} \le V\mathbf{v} \le \mathbf{b}} e^{-\frac{1}{2}\mathbf{v}'\mathbf{v}} d\mathbf{v},$$

where $V = V(0)$. V can be written as $V = \hat{U}D^{\frac{1}{2}}$, where $\hat{U}$ is an $m \times m$ matrix with its first k columns as the first k columns of U and remaining $m-k$ columns as columns of zeros, because $d_{k+1} = d_{k+2} = \ldots = d_m = 0$. For a final step, we determine an $m \times m$ orthogonal matrix Q so that $C = VQ'$ is lower triangular. The Q required for this step is an $m \times m$ identity matrix except for its principal $k \times k$ submatrix which is the $k \times k$ orthogonal matrix required to make the principal $k \times k$ submatrix of V lower triangular (see the book by Golub and Van Loan, 1996, for a description of how such a Q can be determined). Then, if $\mathbf{v} = Q'\mathbf{y}$, $F(\mathbf{a}, \mathbf{b}, \Sigma) = (2\pi)^{-\frac{m}{2}} \int_{\mathbf{a} \le C\mathbf{y} \le \mathbf{b}} e^{-\frac{1}{2}\mathbf{y}'\mathbf{y}} d\mathbf{y}$, where C is lower triangular and $c_{ij} = 0$ if $j > k$. The integration region defined by $\mathbf{a} \le C\mathbf{y} \le \mathbf{b}$ places no constraint on the variables $y_{k+1}, y_{k+2}, \ldots, y_m$, and all possible values between $-\infty$ and ∞ for these variables are consistent with the definition of the integration region, so the innermost $m - k$ integrals are all equal to one. Therefore

$$F(\mathbf{a}, \mathbf{b}, \Sigma) = (2\pi)^{-\frac{k}{2}} \int_{\mathbf{a} \le C\mathbf{y} \le \mathbf{b}} e^{-\frac{1}{2}\mathbf{y}'\mathbf{y}} d\mathbf{y},$$

where C is now the $m \times k$ matrix obtained by removing the original $m-k$ zero columns from the original C and $\mathbf{y} = (y_1, y_2, \ldots, y_k)'$. This matrix C can be computed directly using the generalised Cholesky decomposition algorithm described by Healy (1968).

Following the procedure used by Genz (1992), the next step is to use the lower triangular structure of C to rewrite each of the inequalities $\mathbf{a} \le C\mathbf{y} \le \mathbf{b}$ and explicitly define the integration limits for F to give F in the form

$$F(\mathbf{a}, \mathbf{b}, \Sigma) = (2\pi)^{-\frac{k}{2}} \int_{a_1'}^{b_1'} e^{-\frac{y_1^2}{2}} \int_{a_2'(y_1)}^{b_2'(y_1)} e^{-\frac{y_2^2}{2}} \cdots \int_{a_k'(y_1,\ldots,y_{k-1})}^{b_k'(y_1,\ldots,y_{k-1})} e^{-\frac{y_k^2}{2}} d\mathbf{y},$$

where $a_i'(y_1, \ldots, y_{i-1}) = (a_i - \sum_{j=1}^{i-1} c_{i,j}y_j)/c_{i,i}$ and $b_i'(y_1, \ldots, y_{i-1}) = (b_i - \sum_{j=1}^{i-1} c_{i,j}y_j)/c_{i,i}$, for $i = 1, 2, \ldots, k$. But this definition of F is not complete if $k < m$, because we must take into account the $m - k$ additional constraints $a_i \le \sum_{j=1}^{i} c_{i,j}y_j \le b_i$, for $i = k+1, k+2, \ldots m$, that the k integration variables must satisfy. There are various cases to consider. In order to introduce the general cases, we first consider the case where $c_{ik} \neq 0$ for all $i > k$. In this case we only need to place additional constraints on y_k, and modify the definitions of a_k' and b_k'. We first rewrite the last $m - k + 1$ constraints in the form $a_i - \sum_{j=1}^{k-1} c_{i,j}y_j \le c_{i,k}y_k \le b_i - \sum_{j=1}^{k-1} c_{i,j}y_j$, for $i = k, k+1, \ldots m$. Then we divide these constraints into two groups, the first group consisting of the constraints where $c_{i,k} > 0$ and the second group consisting of the constraints where $c_{i,k} < 0$. For both groups, we divide by $c_{i,k}$ to produce explicit constraints on y_k, but for the second group we must change the order of the inequalities. The revised limits for y_k can now be written as

$$\bar{a}_k'(y_1, \ldots, y_{k-1}) = \max\Big(\max_{c_{i,k}>0} \Big(\frac{a_i - \sum_{j=1}^{k-1} c_{i,j}y_j}{c_{i,k}} \Big), \max_{c_{i,k}<0} \Big(\frac{b_i - \sum_{j=1}^{k-1} c_{i,j}y_j}{c_{i,k}} \Big) \Big),$$

and

$$\bar{b}'_k(y_1,\ldots,y_{k-1}) = \max\Big(\quad \bar{a}'_k(y_1,\ldots,y_{k-1}),$$
$$\min\Big(\min_{c_{i,k}>0}\Big(\frac{b_i - \sum_{j=1}^{k-1} c_{i,j}y_j)}{c_{i,k}}\Big),\ \min_{c_{i,k}<0}\Big(\frac{a_i - \sum_{j=1}^{k-1} c_{i,j}y_j)}{c_{i,k}}\Big)\Big)\Big).$$

In the cases where $c_{i,k} = 0$ for some i's, with $i > k$, the associated constraints for these i's do not affect y_k, so the constraints for some of other variables need to be adjusted. For these general cases, we first reorder all of the constraints into groups of sizes $l_1, l_2, \ldots, l_k$, with $l_1 + l_2 + \cdots + l_k = m$, so that if we denote the reordered constraints by $\bar{\mathbf{a}} \le \bar{C}\mathbf{y} \le \bar{\mathbf{b}}$, then the first l_1 rows of $\bar{C}$ have the form $(*, 0, \ldots, 0)$, the next l_2 rows of $\bar{C}$ have the form $(?, *, 0, \ldots, 0)$, and so on with the last l_k rows of $\bar{C}$ in the form $(?, \ldots, ?, *)$, where $*$ denotes a nonzero and ? denotes zero or nonzero. Then, for each i, $i = 1, \ldots, k$, the set of l_i constraints are rewritten and merged to produce a single constraint on y_i using the procedure that we described for y_k. When this process is complete, the integral for F can be written as

$$F(\mathbf{a},\mathbf{b},\Sigma) = (2\pi)^{-\frac{k}{2}} \int_{\bar{a}'_1}^{\bar{b}'_1} e^{-\frac{y_1^2}{2}} \int_{\bar{a}'_2(y_1)}^{\bar{b}'_2(y_1)} e^{-\frac{y_2^2}{2}} \cdots \int_{\bar{a}'_k(y_1,\ldots,y_{k-1})}^{\bar{b}'_k(y_1,\ldots,y_{k-1})} e^{-\frac{y_k^2}{2}} d\mathbf{y}.$$

In order to put F into a form that is easy to use with standard numerical integration methods, two more transformations are required. First, let $y_i = \Phi^{-1}(z_i)$, for $i = 1, 2, ..k$ so that $\phi(y_i)dy_i = dz_i$. Then

$$F(\mathbf{a},\mathbf{b},\Sigma) = \int_{d_1}^{e_1} \int_{d_2(z_1)}^{e_2(z_1)} \cdots \int_{d_k(z_1,\ldots,z_{k-1})}^{e_k(z_1,\ldots,z_{k-1})} d\mathbf{z},$$

with

$$d_i(z_1,\ldots,z_{i-1}) = \Phi(\bar{a}'_i(\Phi^{-1}(z_1),\ldots,\Phi^{-1}(z_{i-1}))),$$

and

$$e_i(z_1,\ldots,z_{i-1}) = \Phi(\bar{b}'_i(\Phi^{-1}(z_1),\ldots,\Phi^{-1}(z_{i-1}))).$$

Finally, let $z_i = d_i + (e_i - d_i)u_i$, for $i = 1, \ldots, k$, so $dz_i = (e_i - d_i)du_i$. Then

$$F(\mathbf{a},\mathbf{b},\Sigma) = (e_1 - d_1)\int_0^1 (e_2(u_1) - d_2(u_1)) \cdots \int_0^1 (e_k(u_1,\ldots,u_{k-1}) - d_k(u_1,\ldots,u_{k-1})) \int_0^1 d\mathbf{u}.$$

The innermost integral is one, so the numerical problem involves the estimation of a $(k-1)$ dimensional integral.

3 Numerical and Simulation Studies

Assume that Z_j for $j = 1, \ldots, m$ are independently and normally distributed with mean 0 and variance $(1+\alpha_j^2)/\alpha_j^2$. Let $\tilde{Z} = \sum_{j=1}^m Z_j\alpha_j^2/(1+\alpha_j^2)$. Kwong (1995) showed that the standardized multivariate normal random variables with singular correlation structure given in (3) can be generated by the transformation $X_j = \alpha_j(Z_j - \tilde{Z})$ for $j = 1, \ldots, m$, where $\sum_{j=1}^m \alpha_j^2/(1+\alpha_j^2) = 1$. Therefore, for any given b_j and α_j for $j = 1, \ldots, m$, we generate all the Z_j and transform each of them to X_j based on (4). Then, we observe whether absolute value of each X_j is less than its corresponding b_j for $j = 1, \ldots, m$, respectively. The process is repeated N times, and the nominal probabilities from the simulation and a standard error are calculated. Those simulated probabilities are compared with two bounds obtained numerically according to Section 2.1, and with the numerical evaluation of the F integrals described in Section 2.2. Randomized lattice rules (see Cranley and Patterson, 1976), were used for the numerical integration of F, and the absolute accuracy requested was 0.001. For this method, the amount of work required was measured as the number N of integrand values (f values) required to estimate F with error less than 0.001. The error estimates used

for the randomized lattice rules were three times the standard errors for these randomized rules. In order to compare these values with values from the simulation method, we used the same N for the simulation method, and report an error estimate for the simulation that is three times the standard error for the simulation method.

Some selected cases for $m = 4, \ldots, 12$ are presented in Table 1. It is obvious that the differences among the two bounds are negligible in all the considered cases. However, the computational time of evaluating the bounds increases rapidly as m increases. It is impractical to compute the bounds for $m > 12$. The computational time of new approach described in Section 2.2 also increases with m, but the estimate values of all the cases considered in this study were obtained in a short period of computational time. The error estimates for the simulation method, using the same number of function values, were in all cases significantly larger than the error estimates for the new method. The new method can also be applied to the multivariate normal distributions with any arbitrary singular correlation structures. Therefore, we conclude that the proposed approach provides an efficient and accurate way to estimate the F integrals with any singular correlation matrices.

4 Applications

Construction of simultaneous confidence intervals for multinomial proportions is one of the most important statistical procedures in many real-life applications, such as statistical quality control, survey sampling, opinion polls, *etc.* Basically, there are two popular forms of simultaneous confidence intervals for m-dimension multinomial proportions. The first one is the conventional confidence intervals (CCI) which have lengths proportionate to the estimated standard error of sample proportions. Therefore, the length of each interval depends on how much the corresponding sample proportion deviates from 0.5, longer interval for less deviation and shorter interval for more deviation. Gold (1963), Cochran (1963), Quesenberry and Hurst (1964), Goodman (1965), Tortora (1978), Bailey (1980), Kwong and Iglewicz (1996), focused on the construction of CCI. The second one is called quick simultaneous confidence intervals (QSCI) which have same length for all the intervals. Angers (1974), Thompson(1987), Fitzpatrick and Scott (1987), Sison and Glaz (1995), Kwong (1998) discussed the procedures for the construction of QSCI. A brief comparison between the two forms of confidence intervals can be found in Fitzpatrick and Scott (1987).

After incorporating the singular correlation structure based on Kwong's Inequalities into the evaluation of the critical values for the construction of QSCI, Kwong (1998) improved the performance of QSCI. However, the approach works only for $m \leq 12$ as the computational time increases rapidly for large m. With the application of the new approach of evaluating singular multivariate normal distribution described in Section 2.2, we can construct QSCI (see Kwong (1998) for details) for any m. Analogous to the case of construction of QSCI, we modify the existing approaches for constructing CCI after taking the singular correlation structure of sample proportions into consideration. Besides, we extend the results to the construction of all pairwise simultaneous confidence intervals for multinomial proportions in the forms of CCI and QSCI.

4.1 Confidence Intervals for Multinomial Proportions

In a sample of size N, let n_j for $j = 1, 2, \ldots, m$ be the observed frequencies in the j-th cell of a m-cell multinomial distribution with corresponding parameter $\mathbf{p} = (p_1, p_2, \ldots, p_m)'$. The maximum likelihood estimator of $\mathbf{p}$ is $\hat{\mathbf{p}} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_m)'$ where $\hat{p}_j = n_j/N$ for $j = 1, 2, \ldots, m$. Asymptotically, $\sqrt{N}(\hat{\mathbf{p}} - \mathbf{p})$ has m-variate normal distribution with zero mean vector and variance-covariance matrix Σ with elements

$$\begin{aligned} \sigma_{jj} &= p_j(1 - p_j) \quad (j = 1, 2, \ldots, m) \\ \sigma_{jk} &= -p_j p_k. \quad (j \neq k) \end{aligned}$$

Therefore, the corresponding correlation matrix has singular negative product structure with $\{\rho_{jk}^{(m)} = -\rho_j \rho_k\}$, where $\rho_j = \sqrt{p_j/(1 - p_j)}$, for $j = 1, 2, \ldots, m$, and rank $m - 1$.

Table 1: Bounds and Estimated Values for MVN Probabilities

b_j's $\alpha_j^2/(1+\alpha_j^2)$'s	Upper Bound	Lower Bound	f Values Simulated Error Est.	f Values F Estimate Error Est.
(2.3, 2.2, 2.1, 2.0)	.887369	.887310	4224 .888968	4224 .887541
(.2, .1, .4, .3)			.014504	.000374
(.5, 2.4, 1.0, 2.0, 1.6)	.232658	.232373	496 .286290	496 .232567
(.1, .2, .2, .2, .3)			.060951	.000642
(2.2, 2.4, 2.5, 2.0, 2.1)	.880775	.880773	6992 .879720	6992 .878440
(.3, .1, .05, .5, .05)			.011671	.000682
(2.4, .5, 1.2, .4, 1.9, 2.0)	.089252	.089103	496 .066532	496 .089192
(.1, .1, .2, .2, .2, .2)			.033603	.000479
(1.6, 1.7, 1.8, 1.4, 2.1, 2.5, 1.6)	.554366	.554366	6692 .560212	6692 .554429
(.1, .1, .2, .2, .2, .1, .1)			.017809	.000979
(2.0, 2.1, 1.9, 1.8, 2.0, 2.1, 2.2, 2.3)	.714231	.714231	6692 .698227	6692 .713891
(.1, .1, .1, .1, .15, .05, .2, .2)			.016470	.000985
(.4, 2.2, 2.5, 3.1, .9, 1.8, .8, 2.3, 2.9)	.102861	.102861	496 .098790	496 .102832
(.01, .02, .07, .1, .15, .05, .3, .2, .1)			.040234	.000269
(2.8, 2.9, 2.8, 2.7, 2.4, 3.3, 3.4, 2.5, 2.6, 2.7)	.935023	.935023	1248 .927885	1248 .934968
(.1, .05, .05, .04, .06, .1, .15, .15, .1, .2)			.021976	.000658
(3.0, 2.8, 2.4, 2.5, 1.9, 2.2, 2.1, 2.0, 2.4, .9, 1.8)	.475903	.475903	6992 .496281	6992 .475583
(.02, .08, .04, .06, .1, .1, .16, .14, .15, .1, .05)			.017939	.000827
(2.5, 2.7, 3.4, .9, 2.4, 1.7, 1.8, 2.3, 2.4, 2.6, .9, .8)	.185877	.185877	496 .181452	496 .185936
(.01, .03, .06, .05, .05, .1, .15, .05, .1, .14, .16, .1)			.051966	.000689

Gold (1963) and Quesenberry and Hurst (1964) provided two alternative asymptotic simultaneous confidence intervals for $\mathbf{p}$ as follows:

$$p_j \in \hat{p}_j \pm \chi \left[\frac{\hat{p}_j(1-\hat{p}_j)}{N} \right]^{\frac{1}{2}} \tag{4}$$

and

$$p_j \in \frac{\chi^2 + 2n_j \pm \left\{ \chi^2 \left[\chi^2 + 4n_j(N-n_j)/N \right] \right\}^{\frac{1}{2}}}{2(N+\chi^2)} \tag{5}$$

for $j = 1, 2, \ldots, m$, respectively, while Bailey (1980) suggested the other two confidence intervals for $\mathbf{p}$ as

$$p_j \in \left\{ \sin \left[Y_{1j} \pm \frac{\chi}{\sqrt{4N+2}} \right] \right\}^2 \tag{6}$$

and

$$p_j \in \frac{\left\{ Y_{2j} \pm \left[C(C+1-Y_{2j}^2) \right]^{\frac{1}{2}} \right\}^2}{(C+1)^2} \tag{7}$$

for $j = 1, 2, \ldots, m$, where $C = \chi^2/(4N)$, $Y_{1j} = \sin^{-1}\sqrt{(n_j+3/8)/(N+3/4)}$, and $Y_{2j} = \sqrt{(n_j+3/8)/(N+1/8)}$. Based on the Bonferroni inequality, without consideration of the singular correlation structure of $\hat{\mathbf{p}}$, Goodman's (1965) criterion (hereafter called GC) is usually used to determine $\chi^2 = \chi_1^2(\alpha/m)$ for the above four techniques in (4)-(7), where $\chi_1^2(\alpha/m)$ is defined as the $100(1-\alpha/m)$ percentage point of the chi-square distribution with 1 degree of freedom.

After taking into account the singular correlation structure of $\hat{\mathbf{p}}$, Kwong and Iglewicz (1996) proposed another criterion (hereafter called KIC) of setting $\chi^2 = z_m^2(\alpha)$ in (4)-(7),where $z_m(\alpha)$ is defined as the two-sided $100(1-\alpha)$ percentage point of the standardized m-variate normal distribution with the singular negative equi-correlated structure. Kwong and Iglewicz (1996) conducted a simulation study to compare the performances of these two criteria. The study concluded that KIC provides less conservative and shorter confidence intervals than GC does when the sample size is sufficiently large.

However, based on the conjecture that the least favorable parameter vector having all the elements equal to one another, KIC uses only part of singular correlation structure. Intuitively, it is possible to improve KIC if the singular negative product correlation structure, instead of singular equi-correlated structure, is incorporated into the evaluation of critical values.

For a given confidence level $1-\alpha$, let

$$H_m[t_\alpha(\mathbf{p})] \quad = \quad \Pr\left[\bigcap_{j=1}^{m} |X_j| \le t_\alpha(\mathbf{p});\ \left\{ \rho_{jk}^{(m)} = -\rho_i \rho_j \right\} \right] = 1-\alpha.$$

By the central limit theorem,

$$\lim_{N\to\infty} \Pr\left[\bigcap_{j=1}^{m} \frac{\sqrt{N}|\hat{p}_j - p_j|}{\sqrt{p_j(1-p_j)}} \le t_\alpha;\ \left\{ \rho_{jk}^{(m)} = -\rho_i \rho_j \right\} \right] = H_m[t_\alpha(\mathbf{p})].$$

Since the correlation structure of $\{\rho_{jk}^{(m)}\}$ is dependent on the unknown parameter $\mathbf{p}$, the sample proportions are used to replace the parameters in order to estimate the correlation structure. Therefore, based on the result in Section 2.2, we numerically solve

$$H_m[t_\alpha(\hat{\mathbf{p}})] = 1-\alpha.$$

for $t_\alpha(\hat{\mathbf{p}})$ and then use it to construct the simultaneous confidence intervals for multinomial proportions when the sample size is relatively large. The numerical solution of this equation requires the use of a nonlinear

equation solving method. We considered the use of variations on the secant method for this problem because the computation of the derivative of H_m requires the evaluation of m m–1–dimensional integrals, so the use of more rapidly convergent method, like Newton's method, is infeasible. When α is small, the objective function $h(t) = H_m(t) - 1 + \alpha$ is very flat near the values of t where $h(t) \approx 0$, and we found that the ordinary secant method frequently diverged unless the starting values for t were very close to the solution. We therefore tried some modified secant methods, which provide at each iteration a bounding interval for the final solution. One of these methods, called the "Pegasus" method (see Ralston and Rabinowitz, 1978, p. 341), which we found to be robust, reliable and efficient, was used for the results presented in the remainder of this paper.

The Pegasus method requires an initial bounding interval for the solution point t^* ($t^* = t^*_\alpha(\hat{\mathbf{p}})$, with $H(t^*_\alpha(\hat{\mathbf{p}})) = 1 - \alpha$). A crude initial bounding interval can be determined from the simple Bonferroni bound. Using notation similar to that in Section 2.1, we let $A_j(t) = \{X_j : |X_j| \le t\}$, and then $P(\cup_{j=1}^m A_j^c(t)) = 1 - H_m(t)$. If we let $S_1(t) = \sum_{j=1}^m P(A_j^c(t))$, then the simple Bonferroni bound is $P(\cup_{j=1}^m A_j^c(t)) \le S_1(t)$. A simple lower bound is $\min_j P(A_j^c(t)) \le P(\cup_{j=1}^m A_j^c(t))$. Combining these bounds, and using $P(A_j^c(t)) = 2\Phi(-t)$, a crude bounding interval for t^* is given by (t_{CL}, t_{GC}), where $t_{GC} = \Phi^{-1}(1-\alpha/(2m))$ (equivalent to GC previously discussed) and $t_{CL} = \Phi^{-1}(1-\alpha/2)$. When m is large, the repeated evaluation of $h(t)$ requires expensive numerical integrations, so in order to reduce the number of evaluations of $h(t)$, we looked for better bounds that could be used to reduce the size of the initial bounding interval, thereby reducing the number of steps required for the convergence of the Pegasus method. Better bounds can be inexpensively computed from combinations of univariate and bivariate probabilities. The best bounds using these probabilities are the Hunter-Worsley bound for an upper t^* limit (see Hsu, 1996, p. 229), and a modified Bonferroni bound for a lower t^* limit (see Kwerel, 1975). If we let $S_2(t) = \sum_{j<i} P(A_j^c(t) \cap A_i^c(t))$, the Hunter-Worsley and modified Bonferroni bounds guarantee that

$$2(S_1(t) - S_2(t)/k)/(k+1) \le P(\cup_{j=1}^m A_j^c(t)) \le S_1(t) - \sum_{(i,j)\in T^*} P(A_j^c(t) \cap A_i^c(t)),$$

where $k = 1 + 2\lfloor S_2/S_1 \rfloor$, and T^* is a maximal spanning tree for the complete graph of order m with edge weights given by $P(A_j^c(t) \cap A_i^c(t))$. If we use t_{MB} to denote the t value where $2(S_1(t) - S_2(t)/k)/(k+1) = \alpha$ and t_{HW} to denote the t value where $S_1(t) - \sum_{(i,j)\in T^*} P(A_j^c(t) \cap A_i^c(t)) = \alpha$, then $t_{CL} \le t_{MB} \le t^* \le t_{HW} \le t_{GC}$. A combined procedure for determining t^* is to first use the Pegasus method, starting with (t_{CL}, t_{GC}), to determine t_{HW}, then use the Pegasus method, starting with (t_{CL}, t_{HW}), to determine t_{MB}, and finally to use the Pegasus method starting with (t_{MB}, t_{HW}) and combined with numerical integration for computing $h(t)$, to determine t^*. For the tests reported in Table 2, we found that the interval (t_{MB}, t_{HW}) is usually very small, particularly for the smaller α values, and in some cases (depending on the accuracy required for t^*) it was not necessary to complete the final step in this procedure.

Another problem that arises when an iterative numerical method like the Pegasus method is combined with a numerical integration method, is the problem of balancing the errors from the two methods. We let $\hat{t} = t^* + \epsilon_t$ be an approximation to t^*, and $\hat{h(t)} = h(t) + \epsilon_I$ be the numerical integration estimate of $h(t)$. When carrying out the iterations for the Pegasus method, if we attempt to compute $h(t)$ at a point $\hat{t}$ close to t^*, we actually compute

$$\hat{h(t)} = h(\hat{t}) + \epsilon_I \approx \epsilon_t h'(t^*) + \epsilon_I.$$

If we want to be able to stop the hybrid numerical procedure when $\epsilon_t < \delta$ for some small δ, then we need an estimate of $h'(t^*)$ so that we can set the absolute error tolerance for our numerical integration method to be less than $\delta/h'(t^*)$. As the Pegasus iterations are carried out we can compute a simple estimate $\hat{h'}$ for $h'(t^*)$ with a difference quotient. We found $\hat{h'}$ as small as .01 for some problems for the smallest α values considered, but in these cases we usually have very small starting intervals, and often no numerical integrations are necessary. The numerical integration error tolerance was set at $\delta/(10\hat{h'})$ for the tests that we carried out.

Selected test results are summarized in Table 2. We used the same sets of p_j's that were used for the tests that were done to produce Table 1. We also added two other p_j sets for $m = 5$. The error tolerance

δ was set at 0.001 for all of the tests. In the cases where $t_{HW} - t_{MB} < 2\delta$, no numerical integration was necessary, and we set $t^* = (t_{HW} + t_{MB})/2$. For most of these problems, the computation time required to compute the required t^* values to within the $\delta = 0.001$ requested accuracy was not significant. In most cases the modified Bonferroni and Hunter-Worseley bonds could be used to quickly provide t^* values to within the $\delta = 0.01$, and the KIC value was very accurate. The two additional p_j sets were added for $m = 5$ to provide examples where the KIC value is not as accurate. This occurs when there is significant variation in the p_j values.

4.2 Pairwise Confidence Intervals for Multinomial Proportions

Gold (1963) proposed the simultaneous CCI for the $M = m(m-1)/2$ differences $\Delta_{ij} = p_i - p_j$ for $1 \le i < j \le m$ as follows:

$$\Delta_{ij} \in \delta_{ij} \pm \sqrt{\frac{A\hat{d}_{ij}}{N}} \tag{8}$$

where $\delta_{ij} = \hat{p}_i - \hat{p}_j$, $\hat{d}_{ij} = \hat{p}_i + \hat{p}_j - (\delta_{ij})^2$ and $A = \chi^2_{M-1}(\alpha)$. Later, Goodman (1965) used the GC to improve the performance of the approach by re-defining $A = \chi^2_1(\alpha/M)$ in (8). In the case of constructing the pairwise QSCI for Δ_{ij} for $1 \le i < j \le m$, Fitzpatrick and Scott (1987) suggested:

$$\Delta_{ij} \in \delta_{ij} \pm \frac{t}{\sqrt{N}} \tag{9}$$

where $t \ge \sqrt{3}$ is determined by solving the following equation

$$\lim_{N\to\infty} \Pr\left[\bigcap_{1\le i<j\le m} \sqrt{N}|\delta_{ij} - \Delta_{ij}| \le t\right] \ge 1 - 2[1 - \Phi(t)] - 4[m-2][1 - \Phi(t\sqrt{2})] = 1 - \alpha$$

for any given confidence level of $1-\alpha$, where Φ is the standard normal distribution function. However, both approaches are based on the Bonferroni inequality, without any consideration of the singular correlation structure of δ_{ij}. Therefore, the intervals are always wider than what they should be at a given confidence level. In order to get more concise intervals, we incorporate the singular correlation structure of δ_{ij} into the determination of the critical values A and t in (8) and (9), respectively.

Let X_{ij} for $1 \le i < j \le m$ be the standardized M-variate normal random variates with the correlation matrix Σ. For a given confidence level $1-\alpha$, define

$$I_M[b_\alpha(\mathbf{p}), \Sigma(\mathbf{p})] = \Pr\left[\bigcap_{1\le i<j\le m} |X_{ij}| \le b_\alpha;\ \Sigma(\mathbf{p})\right] = 1 - \alpha$$

and

$$J_M[h_\alpha(\mathbf{p}), \Sigma(\mathbf{p})] = \Pr\left[\bigcap_{1\le i<j\le m} |X_{ij}| \le \frac{h_\alpha}{\sqrt{d_{ij}}};\ \Sigma(\mathbf{p})\right] = 1 - \alpha$$

where $d_{ij} = p_i + p_j - \Delta^2_{ij}$ and the correlation matrix $\Sigma(\mathbf{p})$ has the following structure:

$$\begin{aligned}
\mathrm{var}(X_{ij}) &= 1 && (1 \le i < j \le m)\\
\mathrm{corr}(X_{ij}, X_{jl}) &= \frac{d_{il} - d_{ij} - d_{jl}}{2\sqrt{d_{ij}d_{jl}}} && (1 \le i < j < l \le m)\\
\mathrm{corr}(X_{ij}, X_{il}) &= \frac{d_{ij} + d_{il} - d_{jl}}{2\sqrt{d_{ij}d_{il}}} && (1 \le i < j < l \le m)
\end{aligned}$$

Table 2: Bounds and Estimated Values for t_α

p_j's	α	t_{CL}	t_{MB}	t^*	t_{KIC}	t_{HW}	t_{GC}	f-Values
(.2, .1, .4, .3)	.100	1.645	2.189	2.190	2.193	2.208	2.241	30016
	.050	1.960	2.465	2.466	2.468	2.476	2.498	38864
	.010	2.576	3.011	3.013	3.013	3.014	3.023	23520
	.005	2.807	3.219	3.220	3.221	3.221	3.227	0
(.1, .2, .2, .2, .3)	.100	1.645	2.288	2.288	2.289	2.308	2.326	44512
	.050	1.960	2.554	2.554	2.555	2.565	2.576	95200
	.010	2.576	3.084	3.084	3.084	3.086	3.090	46768
	.005	2.807	3.286	3.287	3.287	3.288	3.291	0
(.3, .1, .05, .5, .05)	.100	1.645	2.288	2.288	2.289	2.308	2.326	44512
	.050	1.960	2.554	2.554	2.555	2.565	2.576	95200
	.010	2.576	3.084	3.084	3.084	3.086	3.090	46768
	.005	2.807	3.286	3.287	3.287	3.288	3.291	0
(.1, .1, .2, .2, .2, .2)	.100	1.645	2.362	2.363	2.363	2.383	2.394	38256
	.050	1.960	2.621	2.621	2.621	2.632	2.638	50960
	.010	2.576	3.139	3.139	3.139	3.142	3.144	70336
	.005	2.807	3.339	3.340	3.339	3.340	3.341	0
(.10, .30, .05, .50, .05)	.100	1.645	2.280	2.282	2.289	2.296	2.326	15728
	.050	1.960	2.547	2.549	2.555	2.555	2.576	13712
	.010	2.576	3.079	3.079	3.084	3.080	3.090	0
	.005	2.807	3.282	3.283	3.287	3.283	3.291	0
(.05, .05, .05, .05, .80)	.100	1.645	2.277	2.280	2.289	2.290	2.326	97088
	.050	1.960	2.546	2.547	2.555	2.551	2.576	137872
	.010	2.576	3.079	3.079	3.084	3.080	3.090	0
	.005	2.807	3.283	3.283	3.287	3.283	3.291	0
(.1, .1, .2, .2, .2, .1, .1)	.100	1.645	2.421	2.422	2.422	2.441	2.450	72272
	.050	1.960	2.675	2.675	2.676	2.685	2.690	112208
	.010	2.576	3.185	3.185	3.185	3.187	3.189	18544
	.005	2.807	3.382	3.383	3.382	3.383	3.384	0
(.1, .1, .1, .1, .15, .05, .2, .2)	.100	1.645	2.471	2.472	2.473	2.490	2.498	160144
	.050	1.960	2.721	2.722	2.721	2.730	2.734	154880
	.010	2.576	3.224	3.225	3.224	3.226	3.227	20048
	.005	2.807	3.419	3.419	3.419	3.420	3.421	0
(.01, .02, .07, .1, .15, .05, .3, .2, .1)	.100	1.645	2.514	2.515	2.516	2.531	2.539	33680
	.050	1.960	2.760	2.761	2.761	2.768	2.773	50000
	.010	2.576	3.258	3.259	3.258	3.259	3.261	0
	.005	2.807	3.451	3.451	3.451	3.452	3.452	0
(.1, .05, .05, .04, .06, .1, .15, .15, .1, .2)	.100	1.645	2.553	2.554	2.554	2.570	2.576	55536
	.050	1.960	2.796	2.796	2.796	2.804	2.807	484416
	.010	2.576	3.288	3.289	3.288	3.290	3.291	0
	.005	2.807	3.479	3.480	3.479	3.480	3.481	0
(.02, .08, .04, .06, .1, .1, .16, .14, .15, .1, .05)	.100	1.645	2.586	2.588	2.587	2.604	2.609	691008
	.050	1.960	2.827	2.827	2.827	2.835	2.838	489952
	.010	2.576	3.315	3.316	3.315	3.317	3.317	0
	.005	2.807	3.505	3.505	3.505	3.506	3.506	0
(.01, .03, .06, .05, .05, .1, .15, .05, .1, .14, .16, .1)	.100	1.645	2.617	2.618	2.618	2.634	2.638	501248
	.050	1.960	2.855	2.857	2.855	2.863	2.865	74544
	.010	2.576	3.339	3.340	3.339	3.341	3.341	0
	.005	2.807	3.528	3.529	3.528	3.529	3.529	0

$$\begin{aligned}
\mathrm{corr}(X_{ij}, X_{lj}) &= \frac{d_{ij} + d_{lj} - d_{il}}{2\sqrt{d_{ij} d_{lj}}} \qquad (1 \le i < l < j \le m) \\
\mathrm{corr}(X_{ij}, X_{lk}) &= \frac{d_{ik} + d_{jl} - d_{il} - d_{jk}}{2\sqrt{d_{ij} d_{lk}}} \qquad (i \ne j \ne l \ne k)
\end{aligned}$$

with the rank of $m - 1$. By the central limit theorem, it is straightforward to show

$$\lim_{N \to \infty} \Pr\left[\bigcap_{1 \le i < j \le m} \frac{\sqrt{N} |\delta_{ij} - \Delta_{ij}|}{\sqrt{d_{ij}}} \le b_\alpha \right] = I_M[b_\alpha(\mathbf{p}), \Sigma(\mathbf{p})]$$

and

$$\lim_{N \to \infty} \Pr\left[\bigcap_{1 \le i < j \le m} \sqrt{N} |\delta_{ij} - \Delta_{ij}| \le h_\alpha \right] = J_M[h_\alpha(\mathbf{p}), \Sigma(\mathbf{p})].$$

As $N \to \infty$, $\Sigma(\hat{\mathbf{p}})$ converges to $\Sigma(\mathbf{p})$ with probability 1. Therefore, based on the result in Section 2.2 and the secant method, we numerically solve

$$I_M[b_\alpha(\hat{\mathbf{p}}), \Sigma(\hat{\mathbf{p}})] = 1 - \alpha$$

and

$$J_M[h_\alpha(\hat{\mathbf{p}}), \Sigma(\hat{\mathbf{p}})] = 1 - \alpha$$

for $b_\alpha(\hat{\mathbf{p}})$ and $h_\alpha(\hat{\mathbf{p}})$ and then use them to construct the confidence intervals in (8) and (9), respectively, when the sample size is sufficiently large. In other words, we propose to construct the simultaneous CCI and QSCI for Δ_{ij} for $1 \le i < j \le m$ as follows:

$$\Delta_{ij} \in \delta_{ij} \pm \frac{b_\alpha(\hat{\mathbf{p}}) \sqrt{\hat{d}_{ij}}}{\sqrt{N}} \tag{10}$$

and

$$\Delta_{ij} \in \delta_{ij} \pm \frac{h_\alpha(\hat{\mathbf{p}})}{\sqrt{N}} \tag{11}$$

respectively.

The results, from a test using the same set of test problems that were used for Table 2, are given in Tables 3 and 4. For many of these problems, significant amounts of computation time were required to compute the required b^* or h^* values to within the $\delta = 0.001$ requested accuracy. But in most cases the modified Bonferroni and Hunter-Worseley bounds could be used to quickly provide b^* or h^* values to within the $\delta = 0.05$. If somewhat higher accuracy is needed, the results indicate that numerical integration combined with the transformations described in Section 2.2 can be used to reliably compute the required values.

Table 3: Bounds and Estimated Values for b_α

p_j's	α	b_{CL}	b_{MB}	b^*	b_{HW}	b_{GC}	f-Values
(.2, .1, .4, .3)	.100	1.645	2.237	2.273	2.314	2.394	40240
	.050	1.960	2.534	2.552	2.578	2.638	89568
	.010	2.576	3.094	3.101	3.111	3.144	166896
	.005	2.807	3.303	3.307	3.315	3.341	539792
(.1, .2, .2, .2, .3)	.100	1.645	2.381	2.450	2.518	2.576	302512
	.050	1.960	2.689	2.719	2.764	2.807	595408
	.010	2.576	3.240	3.247	3.268	3.291	2455072
	.005	2.807	3.443	3.461	3.463	3.481	995248
(.3, .1, .05, .5, .05)	.100	1.645	2.293	2.408	2.472	2.576	866944
	.050	1.960	2.574	2.678	2.724	2.807	1013984
	.010	2.576	3.176	3.212	3.238	3.291	8447392
	.005	2.807	3.389	3.416	3.436	3.481	8819360
(.1, .1, .2, .2, .2, .2)	.100	1.645	2.471	2.581	2.666	2.713	503280
	.050	1.960	2.795	2.840	2.900	2.935	894416
	.010	2.576	3.345	3.357	3.384	3.403	4035536
	.005	2.807	3.546	3.553	3.573	3.588	7748224
(.1, .1, .2, .2, .2, .1, .1)	.100	1.645	2.571	2.682	2.782	2.823	985312
	.050	1.960	2.871	2.940	3.007	3.038	2097472
	.010	2.576	3.428	3.445	3.477	3.494	5927952
	.005	2.807	3.626	3.637	3.661	3.675	5017872
(.1, .1, .1, .1, .15, .05, .2, .2)	.100	1.645	2.646	2.764	2.863	2.914	874160
	.050	1.960	2.913	3.016	3.085	3.124	2215392
	.010	2.576	3.489	3.516	3.547	3.570	6376832
	.005	2.807	3.688	3.704	3.729	3.748	17451168
(.01, .02, .07, .1, .15, .05, .3, .2, .1)	.100	1.645	2.621	2.784	2.887	2.991	2560512
	.050	1.960	2.906	3.035	3.110	3.197	4601504
	.010	2.576	3.440	3.535	3.575	3.635	47124288
	.005	2.807	3.641	3.726	3.758	3.810	51350192
(.1, .05, .05, .04, .06, .1, .15, .15, .1, .2)	.100	1.645	2.753	2.896	3.009	3.059	1901840
	.050	1.960	3.029	3.141	3.221	3.261	2505888
	.010	2.576	3.579	3.627	3.668	3.692	42844224
	.005	2.807	3.781	3.813	3.845	3.865	44940768
(.02, .08, .04, .06, .1, .1, .16, .14, .15, .1, .05)	.100	1.645	2.782	2.945	3.059	3.118	2152096
	.050	1.960	3.071	3.187	3.269	3.317	3153488
	.010	2.576	3.608	3.672	3.712	3.743	47119568
	.005	2.807	3.815	3.855	3.888	3.914	139592776
(.01, .03, .06, .05, .05, .1, .15, .05, .1, .14, .16, .1)	.100	1.645	2.813	2.983	3.099	3.172	2277408
	.050	1.960	3.091	3.225	3.308	3.368	5534368
	.010	2.576	3.609	3.705	3.748	3.789	40708720
	.005	2.807	3.822	3.888	3.923	3.957	66337872

Table 4: Bounds and Estimated Values for h_α

p_j's	α	h_{CL}	h_{MB}	h^*	h_{HW}	h_{GC}	f-Values
(.2, .1, .4, .3)	.100	1.366	1.564	1.590	1.610	1.680	4240
	.050	1.628	1.794	1.808	1.820	1.871	12208
	.010	2.140	2.247	2.251	2.255	2.281	35024
	.005	2.332	2.420	2.421	2.426	2.445	52160
(.1, .2, .2, .2, .3)	.100	1.151	1.524	1.560	1.599	1.638	308448
	.050	1.372	1.726	1.742	1.767	1.795	351776
	.010	1.803	2.101	2.105	2.114	2.128	1967024
	.005	1.965	2.242	2.244	2.251	2.261	7606464
(.3, .1, .05, .5, .05)	.100	1.434	1.532	1.563	1.574	1.625	68688
	.050	1.709	1.779	1.790	1.794	1.824	47296
	.010	2.246	2.269	2.270	2.271	2.277	0
	.005	2.447	2.461	2.461	2.462	2.465	0
(.1, .1, .2, .2, .2, .2)	.100	1.040	1.449	1.508	1.549	1.584	141680
	.050	1.240	1.648	1.670	1.696	1.722	300144
	.010	1.629	1.989	1.993	2.004	2.017	918384
	.005	1.775	2.115	2.124	2.125	2.135	700064
(.1, .1, .2, .2, .2, .1, .1)	.100	1.040	1.425	1.476	1.510	1.538	300272
	.050	1.240	1.607	1.625	1.646	1.667	1067536
	.010	1.629	1.925	1.928	1.936	1.946	657440
	.005	1.775	2.045	2.047	2.052	2.059	69584
(.1, .1, .1, .1, .15, .05, .2, .2)	.100	1.040	1.368	1.432	1.466	1.503	390544
	.050	1.240	1.543	1.574	1.595	1.625	350656
	.010	1.629	1.858	1.865	1.873	1.890	1331888
	.005	1.775	1.975	1.979	1.985	1.998	1642416
(.01, .02, .07, .1, .15, .05, .3, .2, .1)	.100	1.151	1.430	1.490	1.513	1.556	556848
	.050	1.372	1.631	1.651	1.663	1.695	778448
	.010	1.803	1.979	1.981	1.987	2.004	431424
	.005	1.965	2.113	2.115	2.118	2.131	1256688
(.1, .05, .05, .04, .06, .1, .15, .15, .1, .2)	.100	.970	1.307	1.373	1.414	1.451	733088
	.050	1.155	1.450	1.508	1.535	1.565	1263296
	.010	1.518	1.772	1.784	1.797	1.814	1509216
	.005	1.655	1.886	1.893	1.902	1.916	3747072
(.02, .08, .04, .06, .1, .1, .16, .14, .15, .1, .05)	.100	.916	1.296	1.356	1.388	1.415	640272
	.050	1.091	1.450	1.478	1.499	1.520	744160
	.010	1.434	1.724	1.729	1.737	1.749	5822080
	.005	1.563	1.825	1.827	1.834	1.842	3643264
(.01, .03, .06, .05, .05, .1, .15, .05, .1, .14, .16, .1)	.100	.916	1.278	1.343	1.374	1.408	669760
	.050	1.091	1.423	1.467	1.486	1.512	1007088
	.010	1.434	1.713	1.721	1.728	1.742	5081504
	.005	1.563	1.816	1.820	1.825	1.836	5114800

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