LINEAR PROGRAMMING and CONVEXITY

Background:

- **Convex Sets**: a subset $S$ of $\mathbb{R}^n$ is **convex** if
  \[ tX + (1 - t)Y \in S \text{ for all } X, Y \in S, \text{ when } 0 \leq t \leq 1. \]
  Examples: sets defined by inequalities.

- **Hyperplanes**: a **hyperplane** is a subset $X$ of $\mathbb{R}^n$ defined by
  \[ X = \{(x_1, x_2, \ldots, x_n) : a_1x_1 + a_2x_2 + \cdots + a_nx_n = b\}, \]
  for real constants $a_1, a_2, \ldots, a_n, b$. 
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• **Half-Spaces:** a **half-space** is a subset $H$ of $\mathbb{R}^n$ defined by

$$H = \{(x_1, x_2, \ldots, x_n) : a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b\},$$

for real constants $a_1, a_2, \ldots, a_n, b$.

• **Vertices:** a **vertex** of a convex set $S$ is a point $V$ that is not the midpoint of any line segment connecting any two distinct points $X, Y \in S$.
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Theorems :

- Half-spaces are convex sets
- Intersections of convex sets are convex sets
- The set of feasible solutions for an LP is a convex set.
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- The basic feasible solutions for a standard form LP are vertices for the convex set of feasible solution for the LP.

- If an LP has a finite optimal objective function value, then the optimal value occurs at at least one basic feasible solution.

Note: if canonical form LP has $m$ equations in $n$ unknowns there could be $\binom{n}{m}$ basic feasible solutions. Simplex algorithm steps follow a sequence of basic feasible solutions which improve the objective until an optimum solution is found.