MY ALGEBRA QUAL

DARYL DEFORD

1. Introduction

I took my oral algebra qual on 12/4/2014. The examiners were Tom Shemanske and Dan Rockmore. The entire exam lasted almost exactly an hour, and we went very rapidly over the material. Although I did fill the blackboard several times, I gave most of my answers orally without writing anything. Because we moved so quickly I am sure that there are questions that I have forgotten here.

2. Linear Algebra

Dan: Give some characterizations of diagonalizability over $\mathbb{R}$.
Dan: How does the minimal polynomial relate to the characteristic polynomial?
Dan: Describe the SVD of a matrix.
Dan: How many non–zero singular values are there?
Dan: Why would you want to compute the SVD (list some applications)?
Dan: How does the SVD relate to dimension reduction?
Dan: Can you write down a formula for the pseudo–inverse?
Dan: How does PCA work?
Dan: How are the random variables related?
Dan: Prove the spectral theorem for symmetric matrices.
Tom: Talk about canonical forms.

3. Group Theory

Tom: Let's talk about $p-$Sylow subgroups of $GL_n(\mathbb{F}_p)$:
Tom: Give an example of such a subgroup.
Tom: What must the eigenvalues of any matrix in the subgroup be?
Tom: What form must its minimal and characteristic polynomials have? (Use Frobenius)
Tom: How many $p-$Sylow subgroups are there?
Tom: How does this relate to $SL_n(\mathbb{F}_p)$?
Tom: Prove it.
Tom: Why are the Sylow subgrous the same size?
Tom: Why is $SL_n(\mathbb{F}_p)$ normal in $GL_n(\mathbb{F}_p)$?
Tom: Does that sequence split?
Tom: Characterize all groups of order 75.
Tom: For which n is $(\mathbb{Z}/n\mathbb{Z})^\times$ cyclic?
Dan: (After I had given an embedding of $\mathbb{Z}/3\mathbb{Z}$ into $GL_2(\mathbb{F}_5)$) What is the generator?
Dan: What are its canonical forms?
Dan: What is a group action?
Dan: What is the structure theorm for group actions?
Dan: Derive the class equation.
Dan: Prove Lagrange's Theorem.
Dan: What is the stabilizer of a group action?
Dan: What are the conjugacy classes in S_n?
Dan: How many of them are there?
Dan: Characterize centralizers in S_n.
Dan: What is the centralizer of a permutation of cycle type (n, n) in S_{2n}.

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Dan: Can you give a description of that group in terms of the wreath product?
Dan: What if it were n two–cycles instead?
Dan: What is the general formula with wreath products?

4. Fields

Tom: Let K/F be a finite separable extension, and let L be the Galois closure.
Tom: Characterize the subgroup of $Gal(L/F)$ that corresponds to K.
Tom: If K/F is not normal, how can we characteize the conjugate fields as subfields of L?
Tom: What does that look like in terms of subgroups?
Tom: Characterize the collection of embeddings K/F in terms of G.
Tom: How does G act on these embeddings?
Tom: What kind of action is that?
Tom: What properties does it have?
Tom: Use that action to prove that the norm of K/F is a map into F.
Tom: Consider the polynomial $x^9 - 125$ over $\mathbb{Q}$.
Tom: Draw an extension diagram showing all intermediate fields and their degrees.
Tom: How do we know that the intersection is trivial?
Tom: For each Galois extension on the diagram, give the isomorphism class of its Galois group.
Tom: Is the group solvable?
Tom: Why must the top extension be cyclic?

5. Rings

Tom: What can you tell me about $x^3 - 49$ over $\mathbb{Q}$?
Tom: What is $\mathbb{Q}[x]/\langle x^3 - 49\rangle$?
Tom: What about $\mathbb{Z}[x]/\langle x^3 - 49\rangle$?
Tom: Prove it.
Tom: Why is it not a field?
Dan: Give three characterizations of a Noetherian ring.
Dan: Give an example of a non–Noetherian ring.
Dan: Define an Artinian ring.
Dan: Give two characterizations each of prime ideals and maximal ideals.
Dan: Give some examples of Euclidean Domains.
Dan: Why are Euclidean rings PIDs?

6. Modules

Dan: What is the universal mapping property of the tensor product of two modules?
Dan: What about module homomorphisms?
Tom: How does this relate to localization?
Tom: What is the universal mapping property of $S^{-1}R$?
Tom: Use this to show that $S^{-1}M \cong S^{-1}R \otimes M$.
Tom: Give four characterizations of a projective module.
Tom: What kind of functor is that?
Tom: Is it left or right exact?
Tom: What can you say about the tensor product of two projective modules?
Tom: Prove it.
Tom: What is a basis for the tensor product of two free modules?

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