A LOWER BOUND IN THE LAW OF THE ITERATED LOGARITHM FOR GENERAL LACUNARY SERIES

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Abstract. We prove a lower bound in a law of the iterated logarithm for sums of the form $\sum_{k=1}^{N} a_k f(n_k x + c_k)$ where f satisfies certain conditions and the n_k satisfy the Hadamard gap condition $\frac{n_{k+1}}{n_k} \geq q > 1$.

1. Introduction

One of the most remarkable achievements of probability theory is the classical Law of the Iterated Logarithm (LIL) due to Kolmogorov [Ko]:

Theorem 1.1. Let $S_m = \sum_{k=1}^{m} X_k$ where $\{X_k\}$ is a sequence of real-valued independent random variables. Let s_m^2 be the variance of S_m. Suppose $s_m \to \infty$ and $|X_m|^2 \leq \dfrac{K_m s_m^2}{\log\log(e^e + s_m^2)}$ for some sequence of constants $K_m \to 0$. Then, almost surely,

$$\limsup_{m\to\infty} \frac{S_m}{\sqrt{2s_m^2 \log\log s_m^2}} = 1.$$

This was first proved for Bernoulli random variables by Khintchine [K] and grew out of the efforts of several authors to determine the exact rate of convergence in Borel's theorem on normal numbers. Although the terms in a lacunary trigonometric series are not independent random variables, it is evidenced by many results in analysis which give central limit theorem type behavior or LILs for lacunary trigonometric series, that they exhibit many of the same properties. For example, Salem and Zygmund [SZ] consider the situation when the X_k in Kolmogorov's theorem are replaced by the functions $a_k \cos(n_k\theta)$ on $[-\pi, \pi]$, where the a_k are real and the n_k are integers satisfying the lacunarity condition: there exists a number q so that

$$\frac{n_{k+1}}{n_k} \geq q > 1 \tag{1.1}$$

for every $k = 1, 2, \dots,$ and obtain an upper bound (≤ 1). This was extended to an upper and lower bound for partial sums of the form $\sum_{k=1}^{N} \exp(2\pi i n_k \theta)$ by

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Erdös and Gál [EG], and later extended to general lacunary trigonometric series by M. Weiss [W].

Takahashi [T1] extends the result of Salem and Zygmund beyond trigonometric functions: Suppose n_k is a lacunary sequence of integers and $f \in$ Lip α, $0 < \alpha \leq 1$, $f(x+1) = f(x)$, for all x, and $\int_0^1 f(x)dx = 0$. Then there exists a constant C depending only on α and q such that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N f(n_k t)}{\sqrt{N \log\log N}} \leq C \quad \text{a.e.}$$

Several authors – Dhompongsa [D], Takahashi [T2], and Peter [P] – have considered versions of this with a gap condition weaker than (1.1).

To state the results of this paper, we need to introduce some notation and terminology. Throughout, a cube $Q \subseteq \mathbb{R}^n$ will be called *dyadic* if it has the form

$$Q = [k_1 2^l, (k_1+1)2^l) \times \ldots \times [k_n 2^l, (k_n+1)2^l)$$

for some $l, k_1, \ldots, k_n \in \mathbb{Z}$; for such a cube Q we say that Q has *sidelength* 2^l. Throughout we will use the notation $|E|$ to denote the Lebesgue measure of a measurable set E.

For $m \in \mathbb{Z}$ we let $\mathcal{F}_m$ denote the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength 2^{-m} and we will let $\mathcal{F}$ denote the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength ≤ 1. By a slight abuse of notation, we will also use $\mathcal{F}_m$ to denote the σ-field generated by the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength 2^{-m}. (The usage will be clear from the context.)

Definition 1.2. If f is a function on $\mathbb{R}^n$ we define the modulus of continuity ω of f as $\omega(f,\delta) = \sup\{|f(x) - f(y)| : |x-y| < \delta\}$. When f is clear from context, we will write $\omega(f,\delta) = \omega(\delta)$. We say that f is Dini continuous if

$$\int_0^1 \frac{\omega(\delta)}{\delta} d\delta < \infty. \tag{1.2}$$

It is easy to see if the integral in (1.2) is finite, then $\int_0^c \omega(\delta)/\delta\, d\delta$ is finite for any $c > 0$.

In [MZ] the authors gave a generalization of the LIL of Takahashi in which the gap condition (1.1) is retained, but the class of functions f is widened:

Theorem 1.3. Suppose f is a Dini continuous function on $\mathbb{R}^n$ with the property that $f(x) = 0$ whenever any coordinate of x is an integer, and $\int_Q f(x)dx = 0$ whenever $Q \in \mathcal{F}_0$. Let $\{n_k\}$ be a sequence of positive numbers satisfying the lacunarity condition $\frac{n_{k+1}}{n_k} \geq q > 1$ and let $\{c_k\}$ be a sequence in $\mathbb{R}^n$. Then there exists a constant C, depending only on n, q, and the quantity $\int_0^1 \omega(\delta)/\delta\, d\delta$, such that for any sequence of real numbers $\{a_k\}$ with $A_m = \sqrt{\sum_{k=1}^m a_k^2} \to \infty$ as $m \to \infty$, we have

$$\limsup_{m\to\infty} \frac{|\sum_{k=1}^m a_k f(n_k x + c_k)|}{\sqrt{A_m^2 \log\log A_m^2}} \leq C \quad a.e.$$

The purpose of this paper is to provide a lower bound in the above result.

Theorem 1.4. Assume that f, n_k, a_k, A_m, and c_k are as in the previous theorem, again with $A_m \to \infty$ as $m \to \infty$. Suppose also that f has the property that there exists a number $c_0 > 0$ such that $\frac{1}{|Q|}\int_Q |f(u)|^2 du > c_0$ for all cubes of sidelength at least 1. Set $M_n = \max_{1\leq k\leq n} |a_k|$ and suppose that $M_n^2 \leq \frac{K_n^2 A_n^2}{\log\log A_n^2}$ for some sequence of numbers $K_n \to 0$ as $n \to \infty$. Then, if q is sufficiently large, there exists a constant c, depending only on n, q, c_0 and the quantity $\int_0^1 \omega(\delta)/\delta\, d\delta$, such that

$$\limsup_{m\to\infty} \frac{\sum_{k=1}^m a_k f(n_k x + c_k)}{\sqrt{A_m^2 \log\log A_m^2}} \geq c \quad a.e.$$

Notice that in both of these theorems we do not assume the n_k are integers, nor do we assume any periodicity of f. We do not know the best possible values of C and c in these inequalities. In the classical LILs, $C = c = 1$, but it seems difficult to obtain such precision here. In the lower bound the so called "Kolmogorov condition" $M_n^2 \leq \frac{K_n A_n^2}{\log\log A_n^2}$ is an essential hypothesis, even in the trigonometric case. (See [BM], pg. 81.) The property that $\frac{1}{|Q|}\int_Q |f(u)|^2 du > c_0$ is also necessary and keeps f from becoming too "sparse" at infinity. For example, consider a function f on $\mathbb{R}$ given by $f(x) = \varepsilon_n \sin(2\pi x)$ for $x \in (-n-1, -n] \cup [n, n+1)$, where $\varepsilon_n \to 0$, say montonically. By Theorem 1.3 (or Salem and Zygmund [SZ]),

$$\limsup_{m\to\infty} \frac{\left|\sum_{k=1}^m \sin 2\pi(2^k x)\right|}{\sqrt{m \log\log m}} \leq C \quad a.e.$$

and thus,

$$\limsup_{m\to\infty} \frac{\left|\sum_{k=1}^m f(2^k x)\right|}{\sqrt{m \log\log m}} = 0 \quad a.e.$$

The latter can be seen by breaking the numerator as $\sum_{k=1}^{2N} + \sum_{k=2N+1}^m$ which gives that the limsup is bounded by $C\varepsilon_{2^{N+1}}$ on $(-\infty, -\frac{1}{2^N}] \cup [\frac{1}{2^N}, \infty)$.

The proof of the Theorem will involve a mix of ideas and techniques from Moore and Zhang [MZ], the study of dyadic martingales, and classical probability theory. In Section 2 we will collect some definitions and lemmas which will be used in the course of the proof. Throughout we will use the convention that C and c represent absolute constants, depending only on q, n and the quantity (1.2), whose value may change from line to line. Sometimes we will need to temporarily track constants and these will be labeled as C_1, C_2, etc.

2. Preliminaries

We record some lemmas.

Lemma 2.1. Let $n_1 < n_2 < \dots$ be an infinite sequence of positive numbers satisfying the lacunarity condition $\frac{n_{k+1}}{n_k} \geq q > 1$, $k = 1, 2, \dots$. If $0 < \alpha < \beta$ then

$$\sum_{\alpha \leq n_k \leq \beta} 1 \leq \frac{\log(\beta q/\alpha)}{\log q}, \tag{2.1}$$

Proof. Let k_0 be defined by the inequality $n_{k_0} < \alpha \leq n_{k_0+1}$ (put $n_0 = 0$) and $i \geq 0$ be defined by the inequality $n_{k_0+i} \leq \beta < n_{k_0+i+1}$. If $i = 0$ then (2.1) is true. If $i \geq 1$ then we have $\beta \geq n_{k_0+i} \geq q^{i-1} n_{k_0+1} \geq q^{i-1}\alpha$. Hence $\beta q/\alpha \geq q^i$ and (2.1) follows immediately. $\square$

Lemma 2.2. Suppose k is a positive integer, $c > 0$. Then

(1) $\sum_{j=k+1}^{\infty} \omega\left(\frac{n_k}{n_j} c\right) \leq \max\{\frac{1}{\log 2}, \frac{1}{\log q}\} \int_0^{\frac{2}{q}c} \frac{\omega(\delta)}{\delta} d\delta$
(2) $\sum_{k=1}^{j-1} \omega\left(\frac{n_k}{n_j} c\right) \leq \max\{\frac{1}{\log 2}, \frac{1}{\log q}\} \int_0^{\frac{2}{q}c} \frac{\omega(\delta)}{\delta} d\delta$
(3) $\sum_{j=k+1}^{\infty} \frac{1}{n_j} \leq \frac{1}{n_k} \frac{1}{q-1}$
(4) $\sum_{k=1}^{j-1} \frac{1}{n_k} \leq \frac{1}{n_1} \frac{q}{q-1}$

Proof.

$$\int_0^{\frac{2}{q}c} \frac{\omega(\delta)}{\delta} d\delta = \int_0^{\frac{2}{q}} \frac{\omega(cs)}{s} ds = \int_{\frac{1}{q}}^{\frac{2}{q}} \frac{\omega(cs)}{s} ds + \sum_{k=1}^{\infty} \int_{\frac{1}{q^{k+1}}}^{\frac{1}{q^k}} \frac{\omega(cs)}{s} ds$$

$$\geq \log 2\, \omega(\frac{1}{q} c) + \sum_{k=1}^{\infty} \log q\, \omega\left(\frac{1}{q^{k+1}} c\right) \geq \min\{\log 2, \log q\} \sum_{k=1}^{\infty} \omega\left(\frac{1}{q^k} c\right).$$

Then

$$\sum_{j=k+1}^{\infty} \omega\left(\frac{n_k}{n_j} c\right) \leq \sum_{k=1}^{\infty} \omega\left(\frac{1}{q^k} c\right) \leq \max\{\frac{1}{\log 2}, \frac{1}{\log q}\} \int_0^{\frac{2}{q}c} \frac{\omega(\delta)}{\delta} d\delta \quad \text{and}$$

$$\sum_{k=1}^{j-1} \omega\left(c\frac{n_k}{n_j}\right) \leq \sum_{k=1}^{j-1} \omega\left(\frac{1}{q^k} c\right) \leq \max\{\frac{1}{\log 2}, \frac{1}{\log q}\} \int_0^{\frac{2}{q}c} \frac{\omega(\delta)}{\delta} d\delta$$

which gives (1) and (2). For (3) we have

$$\sum_{j=k+1}^{\infty} \frac{1}{n_j} = \frac{1}{n_k} \sum_{j=k+1}^{\infty} \frac{n_k}{n_j} \leq \frac{1}{n_k} \sum_{j=1}^{\infty} \frac{1}{q^j} = \frac{1}{n_k} \frac{1}{q-1}.$$

The proof of (4) is similar. □

We will need a lower bound for $\| \sum_{k=1}^{N} a_k f(n_k x + c_k)\|_2$ on $[0,1]^n$. This will be done by squaring and estimating the terms $a_k a_j \int_{[0,1]^n} f(n_k x + c_k) f(n_j x + c_j)dx$. We will use the well-established principle that if, say n_j is much larger than n_k, then $f(n_k x + c_k)$ is roughly constant on cubes where $f(n_j x + c_j)$ has mean value zero, which leads to a small value for the integral.

Lemma 2.3. If $j > k$, then

$$\left|\int_{[0,1]^n} f(n_j x + c_j) f(n_k x + c_k)dx\right| \leq \left(\int_{[0,1]^n} |f(n_j x + c_j)|^2 dx\right)^{\frac{1}{2}} \left(\omega(\frac{\sqrt{n}n_k}{2n_j}) + \frac{\sqrt{2n}\|f\|_\infty}{\sqrt{n_j}}\right).$$

Proof. Recall that $\mathcal{F}_0$ denotes the set of all dyadic cubes in $\mathbb{R}^n$ of sidelength 1. Consider the family of cubes of the form $Q_{j,m} = \frac{1}{n_j}Q_m - \frac{1}{n_j}c_j$, where $Q_m \in \mathcal{F}_0$. Note that $\int_{Q_{j,m}} f(n_j x + c_j)dx = 0$. We say $Q_{j,m}$ is of type I if $Q_{j,m} \subset [0,1]^n$, and $Q_{j,m}$ is of type II if $Q_{j,m} \cap [0,1]^n \neq \emptyset$ and $Q_{j,m} \cap ([0,1]^n)^c \neq \emptyset$. Let $R = (\cup Q_{j,m}) \cap [0,1]^n$, where the union is taken over all type II cubes. Then $|R| \leq 1 - \left(1 - \frac{2}{n_j}\right)^n \leq \frac{2n}{n_j}$. For each type I $Q_{j,m}$, let $a_{j,m}$ denote its center. Then

$$\begin{aligned}
&\left|\int_{[0,1]^n} f(n_k x + c_k) f(n_j x + c_j)dx\right| \leq \\
&\left|\sum_{\substack{\text{type I}\\ Q_{j,m}}} \int_{Q_{j,m}} f(n_k x + c_k) f(n_j x + c_j)dx\right| + \int_R |f(n_k x + c_k) f(n_j x + c_j)|dx \\
&\leq \left|\sum_{\substack{\text{type I}\\ Q_{j,m}}} \int_{Q_{j,m}} (f(n_k x + c_k) - f(n_k a_{j,m} + c_k)) f(n_j x + c_j)dx\right| \\
&\qquad + \left(\int_R |f(n_k x + c_k)|^2 dx\right)^{\frac{1}{2}} \left(\int_R |f(n_j x + c_j)|^2 dx\right)^{\frac{1}{2}} \\
&\leq \sum_{\substack{\text{type I}\\ Q_{j,m}}} \omega(\frac{\sqrt{n}n_k}{2n_j}) \int_{Q_{j,m}} |f(n_j x + c_j)|dx \\
&\qquad + \frac{\sqrt{2n}\|f\|_\infty}{\sqrt{n_j}} \left(\int_{[0,1]^n} |f(n_j x + c_j)|^2 dx\right)^{\frac{1}{2}} \\
&\leq \omega(\frac{\sqrt{n}n_k}{2n_j}) \left(\int_{[0,1]^n} |f(n_j x + c_j)|^2 dx\right)^{\frac{1}{2}}
\end{aligned}$$

$$+\frac{\sqrt{2n}\|f\|_\infty}{\sqrt{n_j}}\left(\int_{[0,1]^n}|f(n_jx+c_j|^2dx\right)^{\frac{1}{2}}.$$

□

Lemma 2.4. $|\int_{[0,1]^n} f(n_jx+c_j)dx| \le 2n\frac{\|f\|_\infty}{n_j}$. More generally, if Q is a dyadic cube of sidelength 2^L then $\frac{1}{|Q|}|\int_Q f(n_jx+c_j)dx| \le 2n\frac{2^L\|f\|_\infty}{n_j}$.

Proof. Using the notation of the previous proof we have:

$$\left|\int_{[0,1]^n} f(n_jx+c_j)dx\right| \le \left|\sum_{\text{type I } Q_{j,m}}\int_{Q_{j,m}} f(n_jx+c_j)dx\right| + \int_R |f(n_jx+c_j|dx$$
$$= 0+\int_R |f(n_jx+c_j)|dx \le |R|\|f\|_\infty \le 2n\frac{\|f\|_\infty}{n_j}.$$

The second statement follows from this by a change of variables. □

Lemma 2.5. If q is sufficiently large, then

$$\int_{[0,1]^n}|\sum_{k=1}^N a_kf(n_kx+c_k)|^2dx \ge cA_N^2$$

for some constant $c>0$ depending only on n, q and the quantity in (1.2).

Proof.

$$\int_{[0,1]^n}\left(\sum_{k=1}^N a_kf(n_kx+c_k)\right)^2dx$$
$$=\sum_{k=1}^N a_k^2\int_{[0,1]^n}|f(n_kx+c_k)|^2dx$$
$$+2\sum_{k=1}^N\sum_{j=k+1}^N a_ka_j\int_{[0,1]^n} f(n_kx+c_k)f(n_jx+c_j)dx$$

For typographical convenience in what follows, set $m_q=\max\{\frac{1}{\log 2},\frac{1}{\log q}\}$. We estimate the second term, using Lemma 2.3 and all parts of Lemma 2.2

$$|\sum_{k=1}^N\sum_{j=k+1}^N a_ka_j\int_{[0,1]^n} f(n_kx+c_k)f(n_jx+c_j)dx|$$
$$\le\sum_{k=1}^N\sum_{j=k+1}^N|a_ka_j|\left(\int_{[0,1]^n}|f(n_jx+c_j)|^2dx\right)^{\frac{1}{2}}\left(\omega(\frac{\sqrt{n}n_k}{2n_j})+\frac{\sqrt{2n}\|f\|_\infty}{\sqrt{n_j}}\right)$$
$$\le\sum_{k=1}^N|a_k|\left(\sum_{j=k+1}^N a_j^2\omega(\frac{\sqrt{n}n_k}{2n_j})\int_{[0,1]^n}|f(n_jx+c_j)|^2dx\right)^{\frac{1}{2}}\left(\sum_{j=k+1}^N\omega(\frac{\sqrt{n}n_k}{2n_j})\right)^{\frac{1}{2}}$$

$$
\begin{aligned}
&+\sqrt{2n}\|f\|_\infty \sum_{k=1}^N |a_k| \left(\sum_{j=k+1}^N a_j^2 \int_{[0,1]^n} |f(n_jx+c_j)|^2 dx\right)^{\frac12} \left(\sum_{j=k+1}^N \frac{1}{n_j}\right)^{\frac12} \\
\le &\left(m_q \int_0^{\frac{\sqrt{n}}{q}} \frac{\omega(\delta)}{\delta} d\delta\right)^{\frac12} \sum_{k=1}^N |a_k| \left(\sum_{j=k+1}^N a_j^2 \omega(\frac{\sqrt{n}n_k}{2n_j}) \int_{[0,1]^n} |f(n_jx+c_j)|^2 dx\right)^{\frac12} \\
&+\left(\sqrt{2n}\|f\|_\infty \frac{1}{\sqrt{q-1}}\right) \sum_{k=1}^N |a_k| \left(\sum_{j=k+1}^N \frac{a_j^2}{n_k} \int_{[0,1]^n} |f(n_jx+c_j)|^2 dx\right)^{\frac12} \\
\le &\left(m_q \int_0^{\frac{\sqrt{n}}{q}} \frac{\omega(\delta)}{\delta} d\delta\right)^{\frac12} A_N \left(\sum_{k=1}^N \sum_{j=k+1}^N a_j^2 \omega(\frac{\sqrt{n}n_k}{2n_j}) \int_{[0,1]^n} |f(n_jx+c_j)|^2 dx\right)^{\frac12} \\
&+\left(\sqrt{2n}\|f\|_\infty \frac{1}{\sqrt{q-1}}\right) A_N \left(\sum_{k=1}^N \sum_{j=k+1}^N \frac{a_j^2}{n_k} \int_{[0,1]^n} |f(n_jx+c_j)|^2 dx\right)^{\frac12} \\
= &\left(m_q \int_0^{\frac{\sqrt{n}}{q}} \frac{\omega(\delta)}{\delta} d\delta\right)^{\frac12} A_N \left(\sum_{j=1}^N a_j^2 \int_{[0,1]^n} |f(n_jx+c_j)|^2 dx \sum_{k=1}^{j-1} \omega(\frac{\sqrt{n}n_k}{2n_j})\right)^{\frac12} \\
&+\left(\sqrt{2n}\|f\|_\infty \frac{1}{\sqrt{q-1}}\right) A_N \left(\sum_{j=1}^N a_j^2 \int_{[0,1]^n} |f(n_jx+c_j)|^2 dx \sum_{k=1}^{j-1} \frac{1}{n_k}\right)^{\frac12} \\
\le &A_N \left(\sum_{j=1}^N a_j^2 \int_{[0,1]^n} |f(n_jx+c_j)|^2 dx\right)^{\frac12} \left(m_q \int_0^{\frac{\sqrt{n}}{q}} \frac{\omega(\delta)}{\delta} d\delta + \frac{\sqrt{2nq}\|f\|_\infty}{\sqrt{n_1}(q-1)}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int_{[0,1]^n} \left|\sum_{k=1}^N a_k f(n_kx+c_k)\right|^2 dx \\
&\ge \sum_{k=1}^N a_k^2 \int_{[0,1]^n} |f(n_kx+c_k)|^2 dx - c_q A_N \left(\sum_{k=1}^N a_k^2 \int_{[0,1]^n} |f(n_kx+c_k)|^2 dx\right)^{\frac12}
\end{aligned}
$$

where $c_q = \left(m_q \int_0^{\sqrt{n}/q} \frac{\omega(\delta)}{\delta} d\delta + \frac{\sqrt{2nq}\|f\|_\infty}{\sqrt{n_1}(q-1)}\right)$. By hypothesis, $\int_{[0,1]^n} |f(n_kx + c_k)|^2 dx > c_0$ for every k, and the lemma follows by taking q sufficiently large (and hence c_q sufficiently small). □

Definition 2.6. Suppose $Q \in \mathcal{F}_0$. A *dyadic martingale* on Q is a sequence of integrable functions $\{g_m\}_{m=0}^\infty$ on Q such that each g_m is $\mathcal{F}_m$ measurable and $g_m = E(g_{m+1}|\mathcal{F}_m)$ for every m. Here $E(g_{m+1}|\mathcal{F}_m)$ denotes the conditional expectation: $E(g_{m+1}|\mathcal{F}_m)(x) = \frac{1}{|Q|}\int_Q g_{m+1} dy$, if $x \in Q \in \mathcal{F}_m$. For $k \ge 1$, set $d_k = g_k - g_{k-1}$, and we also define the square function $Sf_m = (\sum_{k=1}^m E(d_k^2|\mathcal{F}_{k-1}))^{1/2}$.

We will need the following subgaussian estimate for dyadic martingales (see Chang, Wilson and Wolff [CWW]).

Lemma 2.7. If g_m is a dyadic martingale on Q then for each m and every $\lambda > 0$,

$$|\{x \in Q : |g_m(x)| \geq \lambda\}| \leq \exp\left(-\frac{\lambda^2}{2\|Sg_m\|_\infty^2}\right)$$

We would like a similar estimate for sums of the form $\displaystyle\sum_{k=1}^{m} a_k f(n_k x + c_k)$.

Lemma 2.8. Put $f_m(x) = \sum_{k=1}^m a_k f(n_k x + c_k)$ where f is as in the hypotheses of Theorem 1.4. Then there exists constants C and c depending only on q, n and the quantity (1.2) such that

$$|\{x \in [0,1]^n : |f_m(x)| \geq \lambda\}| \leq C \exp\left(-c\frac{\lambda^2}{A_m^2}\right).$$

Proof. By Lemma 2.1 we can break up the sequence n_k into a finite number of sequences each of which has the property that for each $k \geq 1$ there exists exactly one n_k with $2^{k-1} \leq n_k < 2^k$. That is, we may write $f_m = f_{m1} + \cdots + f_{mK}$ for some positive integer K so that each f_{mj} has at most one n_k in each dyadic block $[2^k, 2^{k+1})$. Then since $|\{x \in [0,1]^n : f_m(x) > \lambda\}| \leq \sum_{j=1}^K |\{x \in [0,1]^n : f_{mj} > \frac{\lambda}{K}\}|$, the desired estimate follows if we can get such an estimate for each f_{mj}. In other words, we may assume, without loss of generality, that f_m has only one n_k in each dyadic block $[2^k, 2^{k+1})$. We first also assume that $a_1 = a_2 = 0$. For $m \geq 1$, let $f_m(x) := \sum_{k=3}^{m+2} a_k f(n_k x + c_k)$. Under these conditions, it is shown in [MZ], following the techniques of [CWW], that there exists a family of dyadic martingales $\{g_m^{(j)}\}$, $j = 1, \ldots, N$, and an absolute constant C_1 such that

$$\left| f_{m+2}(x) - \sum_{j=1}^{N} g_m^{(j)}(x) \right| \leq C_1 A_{m+2}$$

$$\text{and for each } \ j, \quad (Sg_m^{(j)}(x))^2 \leq C_1 A_{m+2}^2.$$

Here C_1 and N depend only on the dimension n. Thus, for $\lambda > C_1 A_{m+2}$,

$$|\{x \in [0,1]^n : |f_{m+2}(x)| \geq \lambda\}| \leq \left|\{x \in [0,1] : |\sum_{j=1}^{N} g_m^{(j)}(x)| \geq \lambda - C_1 A_{m+2}\}\right|$$

$$\leq \sum_{j=1}^{N} \left|\{x \in [0,1] : |g_m^{(j)}(x)| \geq \frac{\lambda - C_1 A_{m+2}}{N}\}\right|$$

$$\leq \sum_{j=1}^{N} \exp\left(-c\frac{(\lambda - C_1 A_{m+2})^2}{(Sg_m^{(j)}(x))^2}\right) \leq N \exp\left(-c\frac{(\lambda - C_1 A_{m+2})^2}{C_1^2 A_{m+2}^2}\right)$$

$$\leq C \exp\left(-c\frac{\lambda^2}{A_{m+2}^2}\right).$$

By taking C large enough so that $C\exp(-cC_1^2) \geq 1$, this remains valid for $\lambda \leq C_1 A_{m+2}$. Finally, to remove the assumption that $a_1 = a_2 = 0$, set $\tilde{f}_m(x) = f_m(x) - a_1 f(n_1 x + c_1) - a_2 f(n_2 x + c_2)$, so that $\tilde{f}_m$ satisfies the above inequality. Noting that $\|f\|_\infty \leq C$, where C depends on the quantity in (1.2), and using the inequality $\exp(-c(\alpha-\beta)^2) \leq \exp(-\frac{3c}{4}\alpha^2 + 3c\beta^2)$, valid for $\alpha, \beta > 0$, we have

$$|\{x \in [0,1]^n : |f_m(x)| > \lambda\}| \leq \left|\{x \in [0,1]^n : \tilde{f}_m(x) > \lambda - (|a_1| + |a_2|)\|f\|_\infty\}\right|$$
$$\leq C\exp\left(-c\frac{(\lambda - (|a_1|+|a_2|)\|f\|_\infty)^2}{A_m^2}\right) \leq C\exp\left(-c\frac{\lambda^2}{A_m^2}\right).$$

□

The following is adapted from part of the proof of Proposition 5 in Bañuelos, Klemeš, and Moore [BKM].

Lemma 2.9. Suppose that $g(x)$ is a real valued function defined on a set E, $|E| > 0$, and that

$$\left|\frac{1}{|E|}\int_E g(x)dx\right| \leq \varepsilon A \quad \text{and} \quad \frac{1}{|E|}\int_E g(x)^2 dx \geq c_0 A^2$$

for some constants $A > 0$, $0 < \varepsilon < 1$, $c_0 > 0$. Suppose also that g satisfies

$$|\{x \in E : |g(x)| > \lambda\}| \leq Ce^{-c\frac{\lambda^2}{A^2}}|E| \quad \text{for all} \quad \lambda > 0,$$

where C, c are constants. Then if ε is sufficiently small, there exists a $\delta > 0$, depending only on ε, c_0, C, and c such that

$$|\{x \in E : g(x) \geq \delta A\}| \geq \delta|E|.$$

Proof. Let $0 < \delta < L$ to be chosen momentarily. Then

$$\begin{aligned} c_0 A^2 &\leq \frac{1}{|E|}\int_E |g(x)|^2 dx \\ &= \frac{1}{|E|}\int_{\{x\in E:|g(x)|>LA\}} |g(x)|^2 dx + \frac{1}{|E|}\int_{\{x\in E:|g(x)|\leq LA\}} |g(x)|^2 dx \\ &\leq C(LA)^2 e^{-cL^2} + 2\int_{LA}^\infty \lambda e^{-c\frac{\lambda^2}{A^2}} d\lambda + \frac{LA}{|E|}\int_E |g(x)|dx \\ &\leq CA^2(L^2+1)e^{-cL^2} + \frac{LA}{|E|}\int_E |g(x)|dx \end{aligned}$$

By choosing L sufficiently large, depending on c, C, and c_0, we have

$$C'A \leq \frac{1}{|E|}\int_E |g(x)|dx.$$

But then

$$\frac{1}{|E|}\int_E g^+(x)dx = \frac{1}{2|E|}\int_E |g(x)| + g(x)dx \geq \frac{C'}{2}A - \frac{\varepsilon}{2}A = CA.$$

Thus,

$$CA \leq \frac{1}{|E|}\int_{\{x\in E: g^+\leq \delta A\}} g^+(x)dx + \frac{1}{|E|}\int_{\{x\in E:\delta A<g^+\leq L'A\}} g^+dx$$

$$+\frac{1}{|E|}\int_{\{x\in E:g^+\geq L'A\}} g^+dx \leq \delta A + \frac{L'A}{|E|}|\{x\in E : g^+(x)\geq \delta A\}| + CAL'e^{-c(L')^2}$$

By choosing δ sufficiently small, and L' sufficiently large, the conclusion follows. □

As to be expected, we will need a Borel-Cantelli type lemma for independent, or at least weakly dependent random variables. This is provided by the following, whose proof can be found in Bañuelos and Moore [BM], pg. 79:

Lemma 2.10. For $k = 1, 2, \ldots$, suppose F_k is a collection of dyadic cubes whose union is $[0,1]^n$ such that F_{k+1} is a refinement of F_k. Suppose that the maximum length of the elements of F_k tends to zero. Suppose $\mathcal{E}_k \subset F_k$ has the property:

$$\forall Q \in F_k, \qquad \left| Q \cap \bigcup_{J\in\mathcal{E}_{k+1}} J \right| > |Q|\frac{C}{k}.$$

Set $E_k = \bigcup_{J\in\mathcal{E}_k} J$. Then for a.e. x, $x \in E_k$ i.o.

3. The proof of the theorem

Let M be a fixed large positive number. Define $N_1 \leq N_2 \leq \cdots$ by

$$N_l = \min\left\{ N : \sum_{k=1}^{N} a_k^2 > M^l \right\}.$$

Let $\varepsilon > 0$ and assume $\varepsilon << 1$.

Consider a large positive integer l. Using the definition of N_l and the fact that $|a_{N_l}|^2 < \varepsilon A_{N_l}^2$, for N_l sufficiently large, we can assume that $A_{N_l}^2 = A_{N_l-1}^2 + a_{N_l}^2 < M^l + \varepsilon A_{N_l}^2$ and hence

$$M^l < A_{N_l}^2 < \frac{M^l}{1-\varepsilon}. \tag{3.1}$$

Consequently,

$$(1-\varepsilon)M < \frac{A_{N_{l+1}}^2}{A_{N_l}^2} < \frac{M}{1-\varepsilon}. \tag{3.2}$$

Then by Lemma 2.8 and (3.2) we obtain

$$\left|\{x\in[0,1]^n : |\sum_{k=1}^{N_l} a_k f(n_k x + c_k)| \geq \sqrt{\frac{1+\varepsilon}{cM(1-\varepsilon)}}\sqrt{A^2_{N_{l+1}}\log\log A^2_{N_{l+1}}}\}\right|$$
$$\leq C\exp\left(-c\frac{1+\varepsilon}{cM(1-\varepsilon)}\frac{A^2_{N_{l+1}}\log\log A^2_{N_{l+1}}}{A^2_{N_l}}\right)$$
$$\leq C\exp\left(-\frac{1+\varepsilon}{M(1-\varepsilon)}(1-\varepsilon)M\log\log A^2_{N_{l+1}}\right)$$
$$\leq C\exp\left(-(1+\varepsilon)\log\log M^{l+1}\right) = C((l+1)\log M)^{-(1+\varepsilon)}.$$

So by the Borel-Cantelli lemma, for almost every $x\in[0,1]^n$,

$$\left|\sum_{k=1}^{N_l} a_k f(n_k x + c_k)\right| < \sqrt{\frac{1+\varepsilon}{cM(1-\varepsilon)}}\sqrt{A^2_{N_{l+1}}\log\log A^2_{N_{l+1}}}. \tag{3.3}$$

for all sufficiently large l (depending on x).

The definition of N_l and (3.1) yields:

$$\begin{aligned}\sum_{k=N_l+1}^{N_{l+1}} a_k^2 &= A^2_{N_{l+1}} - A^2_{N_l} > M^{l+1} - \frac{M^l}{1-\varepsilon} = M^{l+1}\left[1-\frac{1}{M(1-\varepsilon)}\right]\\ &\geq A^2_{N_{l+1}}(1-\varepsilon-\frac{1}{M}).\end{aligned} \tag{3.4}$$

By hypotheses, for all sufficiently large l,

$$\max_{1\leq k\leq N_{l+1}} a_k^2 \leq K^2_{N_{l+1}}\left(\frac{A^2_{N_{l+1}}}{\log\log A^2_{N_{l+1}}}\right) \leq \frac{\varepsilon}{2}\left(\frac{A^2_{N_{l+1}}}{\log\log A^2_{N_{l+1}}}\right),$$

which, by (3.4) and the definition of $A_{N_{l+1}}$ implies that

$$\max_{1\leq k\leq N_{l+1}} a_k^2 \leq \frac{K^2_{N_{l+1}}}{1-\varepsilon-\frac{1}{M}}\frac{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}{\log\log A^2_{N_{l+1}}} < \frac{\varepsilon/2}{(1-\varepsilon-\frac{1}{M})}\frac{1}{\log l}\sum_{k=N_l+1}^{N_{l+1}} a_k^2.$$

We may assume that ε is small enough and M large enough so that $1-\varepsilon-\frac{1}{M} > \frac{1}{2}$. Thus,

$$\max_{1\leq k\leq N_{l+1}} \frac{|a_k|}{\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}} \leq \sqrt{\frac{\varepsilon}{\log l}}. \tag{3.5}$$

Let $0<\mu<1$. Suppose l is large so that $\mu\log l >> 1$. We define a sequence of positive integers $l_1, l_2\cdots, l_{\lfloor\rfloor}$, where for simplicity we write $\lfloor\rfloor = \left\lfloor\dfrac{\mu\log l}{1+\varepsilon}\right\rfloor$ ($\lfloor\ \rfloor$ represents the greatest integer function) as follows:

Let l_1 be the first time such that

$$\sum_{k=N_l+1}^{N_l+l_1} a_k^2 \geq \frac{1}{\mu\log l}\sum_{k=N_l+1}^{N_{l+1}} a_k^2,$$

so that

$$\sum_{k=N_l+1}^{N_l+l_1-1} a_k^2 < \frac{1}{\mu \log l} \sum_{k=N_l+1}^{N_{l+1}} a_k^2. \tag{3.6}$$

Likewise, let l_2 be the first time such that

$$\sum_{k=N_l+l_1+1}^{N_l+l_2} a_k^2 \geq \frac{1}{\mu \log l} \sum_{k=N_l+1}^{N_{l+1}} a_k^2,$$

so that

$$\sum_{k=N_l+l_1+1}^{N_l+l_2-1} a_k^2 < \frac{1}{\mu \log l} \sum_{k=N_l+1}^{N_{l+1}} a_k^2. \tag{3.7}$$

Similarly we define $l_3, \ldots, l_{\lfloor\rfloor}$.

Because of (3.6), $N_l + l_1 \leq N_{l+1}$ and hence by (3.6) and (3.5)

$$\sum_{k=N_l+1}^{N_l+l_1} a_k^2 = \sum_{k=N_l+1}^{N_l+l_1-1} a_k^2 + a_{N_l+l_1}^2 \leq \frac{1+\varepsilon}{\mu \log l} \sum_{k=N_l+1}^{N_{l+l}} a_k^2.$$

Combining this and (3.7) yields

$$\sum_{k=N_l+1}^{N_l+l_2-1} a_k^2 \leq \left(\frac{1+\varepsilon}{\mu \log l} + \frac{1}{\mu \log l}\right) \sum_{k=N_l+1}^{N_{l+l}} a_k^2 < \sum_{k=N_l+1}^{N_{l+1}} a_k^2, \tag{3.8}$$

the last inequality being a consequence of the fact that

$$r\left(\frac{1+\varepsilon}{\mu \log l}\right) + \frac{1}{\mu \log l} < 1 \quad \text{for positive integers } r \text{ with } r \leq \left\lfloor \frac{\mu \log l}{1+\varepsilon} \right\rfloor - 1. \tag{3.9}$$

Thus, $N_l + l_2 \leq N_{l+1}$, so by (3.8) and again using (3.5), we have

$$\begin{aligned} \sum_{k=N_l+1}^{N_l+l_2} a_k^2 &= \sum_{k=N_l+1}^{N_l+l_2-1} a_k^2 + a_{N_l+l_2}^2 \leq \left(\frac{1+\varepsilon}{\mu \log l} + \frac{1}{\mu \log l} + \frac{\varepsilon}{\mu \log l}\right) \sum_{k=N_l+1}^{N_{l+l}} a_k^2 \\ &= 2\left(\frac{1+\varepsilon}{\mu \log l}\right) \sum_{k=N_l+1}^{N_{l+l}} a_k^2. \end{aligned} \tag{3.10}$$

Continuing in the same fashion, using (3.5) and (3.9) we have

$$\sum_{k=N_l+1}^{N_l+l_3-1} a_k^2 \leq \left(2\left(\frac{1+\varepsilon}{\mu \log l}\right) + \frac{1}{\mu \log l}\right) \sum_{k=N_l+1}^{N_{l+1}} a_k^2 < \sum_{k=N_l+1}^{N_{l+1}} a_k^2, \tag{3.11}$$

which implies that $N_l + l_3 \leq N_{l+1}$. We continue this process, repeatedly using (3.5) and (3.9) to conclude $N_l + l_{\lfloor\rfloor} \leq N_{l+1}$.

Consider a dyadic cube Q such that $|Q| = 2^{-L}$ where L is chosen so that $2^L \le n_{N_l} < 2^{L+1}$. By rescaling to Q, Lemma 2.5 implies that

$$\int_Q \left| \sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) \right|^2 dx \ge c|Q| \sum_{k=N_l+1}^{N_l+l_1} a_k^2.$$

Similarly, again by rescaling to Q, Lemma 2.8 implies that

$$\left| \{x \in Q : |\sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k)| \ge \lambda\} \right| \le C \exp\left(-c \frac{\lambda^2}{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \right) |Q|.$$

Finally, notice that for k with $N_l + 1 \le k \le N_l + l_1$, (3.5) yields

$$|a_k| \le \sqrt{\frac{\varepsilon}{\log l}} \sqrt{\sum_{k=N_l+1}^{N_{l+l}} a_k^2} \le \sqrt{\frac{\varepsilon}{\log l}} \sqrt{\mu \log l} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} = \sqrt{\mu\varepsilon} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}.$$

Consequently by Lemma 2.4, and Lemma 2.2 (3),

$$\left| \frac{1}{|Q|} \int_Q \sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) dx \right| \le \sum_{k=N_l+1}^{N_l+l_1} |a_k| \frac{2n2^L \|f\|_\infty}{n_k}$$
$$\le \|f\|_\infty \sqrt{\mu\varepsilon} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \sum_{k=N_l+1}^{N_l+l_1} \frac{2n2^L}{n_k} \le C\sqrt{\varepsilon} \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2}.$$

Then Lemma 2.9 applies to give $\delta > 0$ (which depends only on ε and constants that themselves depend only on q and n) so that

$$\left| \left\{ x \in Q : \sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) > \frac{\delta}{\sqrt{\mu \log l}} \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2} \right\} \right|$$
$$\ge \left| \left\{ x \in Q : \sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) > \delta \sqrt{\sum_{k=N_l+1}^{N_l+l_1} a_k^2} \right\} \right| \ge \delta |Q|. \tag{3.12}$$

Set $h(x) = \sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k)$. Choose L_1 so that $2^{L_1} \le n_{N_l+l_1} < 2^{L_1+1}$. Fix $x,\ y$ and suppose $|x - y| < \frac{\sqrt{n}}{2^{L_1}}$. Then using the hypotheses of the theorem, the definition of $A_{N_{l+1}}$, Lemma 2.2 (2) and (3.4), and again assuming that $1-\varepsilon-\frac{1}{M}>\frac{1}{2}$, we have

$$|h(x)-h(y)| \leq \sum_{k=N_l+1}^{N_l+l_1} |a_k|\,|f(n_kx+c_k)-f(n_ky+c_k)|$$

$$\leq \sum_{k=N_l+1}^{N_l+l_1} |a_k|\omega\left(\frac{\sqrt{n}n_k}{2^{L_1}}\right) \leq \frac{K_{N_{l+1}}A_{N_{l+1}}}{\sqrt{\log\log A^2_{N_{l+1}}}} \sum_{k=N_l+1}^{N_l+l_1} \omega\left(\frac{\sqrt{n}n_k}{n_{N_l+l_1}}2\right) \tag{3.13}$$

$$\leq CK_{N_{l+1}}\frac{\sqrt{2\sum_{k=N_l+1}^{N_{l+1}} a_k^2}}{\sqrt{\log l}}.$$

Thus, if $h(x) > \dfrac{\delta}{\sqrt{\mu\log l}}\sqrt{\displaystyle\sum_{k=N_l+1}^{N_{l+1}} a_k^2}$, then

$$|h(y)| \geq |h(x)| - C\frac{K_{N_{l+1}}}{\sqrt{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+l}} a_k^2} \geq \left(\frac{\delta - C\sqrt{\mu}K_{N_{l+1}}}{\sqrt{\mu\log l}}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}.$$

From (3.12) we conclude that there exists a collection of dyadic subcubes $\{Q^{'}\}$ of Q with each $|Q^{'}| = 2^{-L_1}$ such that $\forall x\in Q^{'}$,

$$\sum_{k=N_l+1}^{N_l+l_1} a_kf(n_kx+c_k) \geq \left(\frac{\delta - C\sqrt{\mu}K_{N_{l+1}}}{\sqrt{\mu\log l}}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2},$$

and with $\left|\displaystyle\bigcup_{Q^{'}\subset Q} Q^{'}\right| > \delta|Q|$.

Consider such a $Q^{'}$. Arguing as above we have

$$\left|\left\{x\in Q^{'}: \sum_{k=N_l+l_1+1}^{N_l+l_2} a_kf(n_k+c_k) > \frac{\delta}{\sqrt{\mu\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}\right\}\right|$$

$$\geq \left|\left\{x\in Q^{'}: \sum_{k=N_l+l_1+1}^{N_l+l_2} a_kf(n_k+c_k) > \delta\sqrt{\sum_{k=N_l+l_1+1}^{N_l+l_2} a_k^2}\right\}\right| \geq \delta|Q^{'}|.$$

As previously, this leads us to a collection of dyadic subcubes $\{Q^{''}\}$ of $Q^{'}$ with $|Q^{''}| = 2^{-L_2}$, where L_2 satisfies $2^{L_2}\leq n_{N_l+l_2} < 2^{L_2+1}$, such that $\forall x\in Q^{''}$,

$$\sum_{k=N_l+l_1+1}^{N_l+l_2} a_kf(n_kx+c_k) \geq \left(\frac{\delta - C\sqrt{\mu}K_{N_{l+1}}}{\sqrt{\mu\log l}}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}$$

and with $\left|\bigcup_{Q''\subset Q'} Q''\right| > \delta|Q'|$. We continue this process. Eventually we come to a subcollection of cubes $\{I\}$ with $|I| = 2^{-L_{\lfloor\rfloor}}$, where $\lfloor\rfloor = \left\lfloor\frac{\mu\log l}{1+\varepsilon}\right\rfloor$, and $L_{\lfloor\rfloor}$ is the number satisfying $2^{L_{\lfloor\rfloor}} \le n_{N_l+l_{\lfloor\rfloor}} < 2^{L_{\lfloor\rfloor}+1}$, such that $\forall x \in I$,

$$\sum_{k=N_l+l_{\lfloor\rfloor-1}+1}^{N_l+l_{\lfloor\rfloor}} a_k f(n_k x + c_k) \ge \left(\frac{\delta - C\sqrt{\mu}K_{N_{l+1}}}{\sqrt{\mu\log l}}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}.$$

Moreover, $\left|\bigcup_{I\subset\tilde{Q}} I\right| > \delta|\tilde{Q}|$ where $\tilde{Q}$ is the previous generation cube. On each I, we need to estimate the remaining terms $\sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} a_k f(n_k x + c_k)$. Using (3.5) and Lemma 2.4 we have:

$$\left|\frac{1}{|I|}\int_I \sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} a_k f(n_k x + c_k)dx\right| \le \sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} |a_k| \left|\frac{1}{|I|}\int_I f(n_k x + c_k)dx\right|$$

$$\le C\sqrt{\frac{\varepsilon}{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2} \sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} \frac{2^{L_{\lfloor\rfloor}}\|f\|_\infty}{n_k} \le C_1\sqrt{\frac{\varepsilon}{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}$$

By Chebyshev,

$$\left|\left\{x \in I : \left|\sum_{k=N_l+l_{\lfloor\rfloor]}+1}^{N_{l+1}} a_k f(n_k x + c_k)\right| > 2C_1\sqrt{\frac{\varepsilon}{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}\right\}\right| \le \frac{1}{2}|I|,$$

so that in particular,

$$\sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} a_k f(n_k x + c_k) > -2C_1\sqrt{\frac{\varepsilon}{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2} \tag{3.14}$$

on at least $\frac{1}{2}$ of the measure of I. Choose $\tilde{L}$ so that $2^{\tilde{L}} \le n_{N_{l+1}} < 2^{\tilde{L}+1}$. Let $h(x) = \sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k f(n_k x + c_k)$.

Let x be a point at which (3.14) holds and suppose $|x - y| \le 2^{-\tilde{L}}$. Estimating as before (as in (3.13)) we have:

$$|h(x) - h(y)| \le CK_{N_{l+1}} \frac{\sqrt{2\sum_{k=N_l+1}^{N_{l+1}} a_k^2}}{\sqrt{\log l}}$$

Thus, if

$$h(x) > -2C_1\sqrt{\frac{\varepsilon}{\log l}}\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}$$

then

$$\begin{aligned} h(y) &> \left(-2C_1\sqrt{\frac{\varepsilon}{\log l}} - \frac{CK_{N_{l+1}}}{\sqrt{\log l}}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2} \\ &= -C\left(\frac{\sqrt{\varepsilon}+K_{N_{l+1}}}{\sqrt{\log l}}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}. \end{aligned}$$

Consequently, there exists a collection of dyadic subcubes $\{J\}$ of I with $|J| = 2^{-\tilde{L}}$ such that for every $x \in J$,

$$\sum_{k=N_l+l_{[]}+1}^{N_{l+1}} a_k f(n_k x + c_k) > -C\left(\frac{\sqrt{\varepsilon}+K_{N_{l+1}}}{\sqrt{\log l}}\right)\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2},$$

and with $|\cup_{J\subset I} J| \geq \frac{1}{2}|I|$.

Finally, adding the estimates from all of the above generations, we have

$$\begin{aligned} &\sum_{k=N_l+1}^{N_l+l_1} a_k f(n_k x + c_k) + \cdots + \sum_{k=N_l+l_{\lfloor\rfloor}-1}^{N_l+l_{\lfloor\rfloor}} a_k f(n_k x + c_k) + \sum_{k=N_l+l_{\lfloor\rfloor}+1}^{N_{l+1}} a_k f(n_k x + c_k) \\ &> \left[\left\lfloor\frac{\mu\log l}{1+\varepsilon}\right\rfloor\left(\frac{\delta - C\sqrt{\mu}K_{N_{l+1}}}{\sqrt{\mu\log l}}\right) - C\left(\frac{\sqrt{\varepsilon}+K_{N_{l+1}}}{\sqrt{\log l}}\right)\right]\sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2}. \end{aligned}$$

on a subcollection $\{J\}$ of dyadic subcubes of Q with

$$|Q\cap\bigcup J| > \frac{|Q|}{2}\delta^{\lfloor\frac{\mu\log l}{1+\varepsilon}\rfloor} \geq \frac{|Q|}{2}\delta^{\frac{\mu\log l}{1+\varepsilon}} = \frac{|Q|}{2}e^{(\log\delta)\frac{\mu\log l}{1+\varepsilon}} = \frac{|Q|}{2}l^{\frac{\mu\log(\delta)}{1+\varepsilon}} \geq \frac{|Q|}{2l},$$

where the latter inequality holds if μ is chosen sufficiently small. We remark that neither δ nor ε depend on μ so this is possible.

We may also assume that l is large enough so that

$$\left\lfloor\frac{\mu\log l}{1+\varepsilon}\right\rfloor\Big/\left(\frac{\mu\log l}{1+\varepsilon}\right) > \frac{1}{1+\varepsilon}. \tag{3.15}$$

Thus, on the subcubes J, if l is sufficiently large, we can estimate

$$\begin{aligned}
&\sum_{k=N_l+1}^{N_{l+1}} a_k f(n_k x + c_k) \\
&> \left[\left\lfloor \frac{\mu \log l}{1+\varepsilon} \right\rfloor \left(\frac{\delta - C\sqrt{\mu} K_{N_{l+1}}}{\sqrt{\mu \log l}} \right) - C \left(\frac{\sqrt{\varepsilon} + K_{N_{l+1}}}{\sqrt{\log l}} \right)\right] \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2} \\
&\geq \left[\frac{1}{1+\varepsilon} \frac{\mu \log l}{1+\varepsilon} \left(\frac{\delta - C\sqrt{\mu} K_{N_{l+1}}}{\sqrt{\mu \log l}} \right) - C \left(\frac{\sqrt{\varepsilon} + K_{N_{l+1}}}{\sqrt{\log l}} \right)\right] \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2} \\
&> \eta \sqrt{\log l} \sqrt{\sum_{k=N_l+1}^{N_{l+1}} a_k^2},
\end{aligned}$$

where η depends only on μ, ε, and δ, but can be taken as a fixed positive number for all l sufficiently large. Thus, if we let F_l denote the family of dyadic cubes Q in $[0,1]$ of sidelength 2^{-L} (recall $2^L \leq n_{N_l} < 2^{L+1}$) and let $\mathcal{E}_{l+1}$ denote the union of those cubes J of sidelength $2^{-\tilde{L}}$ (recall $2^{\tilde{L}} \leq n_{N_{l+1}} < 2^{\tilde{L}}$) found in all of the Q using the above argument, then, for large enough l (depending only on ε and M), the hypotheses of Lemma 2.10 are satisfied, so that there exits $\eta > 0$ such that for a.e. x there exists a subsequence of $\{N_l\}_{l=1}^{\infty}$, (depending on x) such that for each l in this subsequence we have

$$\frac{\sum_{k=N_l+1}^{N_{l+1}} a_k f(n_k x + c_k)}{\sqrt{\log l \sum_{k=N_l+1}^{N_{l+1}} a_k^2}} > \eta.$$

For such an x, then by (3.4), and again assuming that $1 - \varepsilon - \frac{1}{M} > \frac{1}{2}$, for an infinite subsequence of the N_l we have

$$\frac{\sum_{k=N_l+1}^{N_{l+1}} a_k f(n_k x + c_k)}{\sqrt{\log l \sum_{k=1}^{N_{l+1}} a_k^2}} > \frac{\eta}{\sqrt{2}}.$$

By (3.1),

$$\log\log A_{N_{l+1}}^2 \leq \log((l+1)\log M - \log(1-\varepsilon)) \leq 2\log l,$$

the latter inequality holding for l sufficiently large. Consequently,

$$\frac{\sum_{k=1}^{N_{l+1}} a_k f(n_k x + c_k) - \sum_{k=1}^{N_l} a_k f(n_k x + c_k)}{\sqrt{\sum_{k=1}^{N_{l+1}} a_k^2 \log\log \left(\sum_{k=1}^{N_{l+1}} a_k^2 \right)}} \geq \frac{\eta}{2}$$

But from (3.3) for a.e. x we have,

$$\frac{\sum_{k=1}^{N_{l+1}} a_k f(n_k x + c_k)}{\sqrt{\sum_{k=1}^{N_l+1} a_k^2 \log\log \sum_{k=1}^{N_l+1} a_k^2}} \leq \sqrt{\frac{1+\varepsilon}{cM(1-\varepsilon)}}$$

for sufficiently all large l (depending on x).

Hence for a.e. x there is an infinite subsequence of sufficiently large enough l so that,

$$\frac{\sum_{k=1}^{N_{l+1}} a_k f(n_k x + c_k)}{\sqrt{\sum_{k=1}^{N_{l+1}} a_k^2 \log\log \sum_{k=1}^{N_{l+1}} a_k^2}} \geq \frac{\eta}{2} - \sqrt{\frac{1+\varepsilon}{cM(1-\varepsilon)}}.$$

Thus, for a.e. x,

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} a_k f(n_k x + c_k)}{\sqrt{\sum_{k=1}^{n} a_k^2 \log\log \sum_{k=1}^{n} a_k^2}} \geq \frac{\eta}{2} - \sqrt{\frac{1+\varepsilon}{cM(1-\varepsilon)}}.$$

We can let $M \nearrow \infty$ and obtain the desired result.

4. Recurrence for partial sums

For $x \in [0,1]^n$, consider the partial sums $s_N(x) = \displaystyle\sum_{k=1}^{N} a_k f(n_k + c_k)$. As a corollary, we show that with an additional hypothesis, for a.e. x, the seqeunce $\{s_N(x)\}$ is dense in $\mathbb{R}$; in other words, this sequence visits every neighborhood of every real number infinitely often. This is a generalization of the same result for lacunary trigonometric series due to Grubb and Moore [GM].

Corollary 4.1. Set $s_N(x) = \sum_{k=1}^{N} a_k f(n_k + c_k)$ where f, a_k, n_k, and c_k are as in the statement of Theorem 1.4. Suppose also that there exists a constant M such that $|a_k| \leq M$ for every k. Then for a.e. $x \in [0,1]^n$, $\{s_N(x)\}$ is dense in $\mathbb{R}$.

The proof follows that of [GM] although with enough differences to justify including the proof here. We first need a variation of a lemma from [GM].

Lemma 4.2. Suppose that two sequences of sets E_N and F_N contained in $[0,1]^n$ have the following property: There exists a constant $c > 0$ and a sequence $\delta_N > 0$ converging to 0 such that for every $x \in E_N$ there is a cube Q_N of sidelength δ_N containing x with $|Q_N \cap F_N| > c|Q_N|$. Suppose that for a.e. $x \in [0,1]^n$, $x \in E_N$ infinitely often. Then for a.e. $x \in [0,1]^n$, $x \in F_N$ infinitely often.

Proof. If we assume the contrary, then there exists a set $A \subset [0,1]^n$ with $|A| > 0$ and a K such that $A \cap (\cup_{N=K}^{\infty} F_N)$ is empty. Almost all points of A are points of density of A so we can pick a point x which is both a point of density of A and in infinitely many E_N. But then, for each $N \geq K$, and

Q_N that contains x, $|Q_N \cap A| \leq |Q_N \cap F_N^c|$ so that as $\delta_N \to 0$, $\frac{|Q_N \cap F_N^c|}{|Q_N|} \geq \frac{|Q_N \cap A|}{|Q_N|} \to 1$, which contradicts the hypothesis. □

Proof. To prove the corollary, let $a \in \mathbb{R}$ and let $\varepsilon > 0$. As an immediate consequence of the theorem, $\sup s_N(x) = \infty$ and $\inf s_N(x) = -\infty$. Thus, a.e. $x \in [0,1]^n$ is in an infinite number of the sets

$$E_N = \{x \in [0,1]^n : s_N(x) \geq a \text{ and } \ s_{N+1}(x) < a\}.$$

We will establish the conditions of the lemma with the sets

$$F_N = \{x \in [0,1]^n : |s_N(x) - a| < \varepsilon \ \text{ or } \ |s_{N+1}(x) - a| < \varepsilon\}.$$

Let $x \in E_N$. Let Q_N be the cube containing x of the form $Q_N = Q_{N+1,m} = \frac{1}{n_{N+1}} Q_m - \frac{1}{n_{N+1}} c_{N+1}$, where $Q_m \in \mathcal{F}_0$ (as in the proof of Lemma 2.3). We first note that if $z, y \in Q_N$ with $|z - y| \leq c/n_{N+1}$, then by Lemma 2.2 (2),

$$\begin{aligned} |s_N(z) - s_N(y)| &\leq \sum_{k=1}^{N} |a_k||f(n_k z + c_k) - f(n_k y + c_k)| \\ &\leq M \sum_{k=1}^{N} \omega\left(\frac{c n_k}{n_{N+1}}\right) \leq M \max\left\{\frac{1}{\log 2}, \frac{1}{\log q}\right\} \int_0^{\frac{2c}{q}} \frac{\omega(\delta)}{\delta} d\delta < \varepsilon, \end{aligned} \tag{4.1}$$

where the last inequality holds if c is sufficiently small. Also

(4.2)

$$\begin{aligned} |s_{N+1}(z) - s_{N+1}(y)| &\leq \sum_{k=1}^{N+1} |a_k||f(n_k z + c_k) - f(n_k y + c_k)| \\ &\leq M \sum_{k=1}^{N+1} \omega\left(\frac{c n_k}{n_{N+1}}\right) \leq M \max\left\{\frac{1}{\log 2}, \frac{1}{\log q}\right\} \int_0^{\frac{2c}{q}} \frac{\omega(\delta)}{\delta} d\delta + M\omega(c) < \varepsilon, \end{aligned}$$

where again, the last inequality holds if c is chosen sufficiently small. Fix $c > 0$ so (4.1) and (4.2) hold. Since $s_N(x) > a$, there are two cases.

Case I: $s_N(x_0) = a$ for some x_0 in Q_N. Then by (4.1) there exists a ball B of radius c/n_{N+1} centered at x_0 such that $B \subset F_N$. At least $\frac{1}{2^n}$ of this ball is in Q_N. Therefore, $|F_N \cap Q_N| > \frac{1}{2^n}|B| = c|Q_N|$.

Case II: $s_N(x) > a$ on Q_N. Since $x \in E_N$, we have $s_{N+1}(x) < a$. But $s_{N+1}(x) = s_N(x) + a_{N+1} f(n_{N+1}x + c_{N+1})$, and $\int_{Q_N} f(n_{N+1}x + c_{N+1}) dx = 0$, so there exists an $x_1 \in Q_N$ where $f(n_{N+1}x_1 + c_{N+1}) = 0$. Consequently, $s_{N+1}(x_1) = s_N(x_1) > a$, so that there exists an $x_0 \in Q$ so that $s_{N+1}(x_0) = a$. By (4.2) there exists a ball B of radius c/n_{N+1} centered at x_0 such that $B \subset F_N$. At least $\frac{1}{2^n}$ of this ball is in Q_N. Again, we have $|F_N \cap Q_N| > \frac{1}{2^n}|B| = c|Q_N|$.

Applying the lemma, we see that for a.e. $x \in [0,1]^n$, $|s_N(x) - a| < \varepsilon$ infinitely often. By considering all a rational and a countable sequence of $\varepsilon \to 0$, the corollary follows.

□

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