MATH 464 – Lecture 5 (01/24/2023)

Today:
* Absolute values in LP
* Two LP formulations (w/ absolute value terms)
* Solving LPs in 2D

Recall

$$\begin{aligned} \min \ & \max_{i=1,\dots,m} (\bar{c}_i^T \bar{x} + d_i) \\ \text{s.t.} \ & A\bar{x} \geq \bar{b} \end{aligned}$$

$$\boxed{\begin{aligned} \min \ & z \\ \text{s.t.} \ & z \geq \bar{c}_i^T \bar{x} + d_i, \ i=1,\dots,m \\ & A\bar{x} \geq \bar{b} \end{aligned}}$$

The intuition one wants to have (for the objective function) is the following. $\max_{i=1,\dots,m} \{\bar{c}_i^T\bar{x}+d_i\}$ represents the surface of a PL "vessel opened upwards". When we compute $\min \max_{i=1,\dots,m} \{\bar{c}_i^T\bar{x}+d_i\}$, we are pushing a blanket or a piece of cloth down on this vessel. We will go down and hit one of the linear pieces. But if we are working with $\max \max_{i=1,\dots,m} \{\bar{c}_i^T\bar{x}+d_i\}$, we would be pulling the blanket up, while the restraining vessel is below it. As such, we could pull the blanket up as much as we want – in fact, the blanket will no longer model the max of the m pieces – it'll just be pulled up without limit!

Similarly, if we want to model $\min_{i=1,\dots,m} f_i(\bar{x}) \leq h$, we do not want to write $f_i(\bar{x}) \leq h \ \forall i$. We only want the smallest of the m functions $f_i(\bar{x})$ to be $\leq h$, while the others could indeed be $> h$. Of course, if each $f_i(\bar{x}) \leq h$, then so will be the smallest of them. But we are putting too much restriction here!

Indeed, we would need extra binary variables to model these situations correctly!

LPs with absolute values

We now consider some direct applications of these ideas. Consider the following optimization problem, which is not an LP as written.

$$\begin{aligned} \min \ & \sum_{i=1}^{n} c_i |x_i| \quad \text{with } c_i \geqslant 0 \\ \text{s.t.} \ & A\bar{x} \geqslant \bar{b} \end{aligned}$$

Recall that $|x_i| = \max\{x_i, -x_i\}$.

If $c_i < 0$, then we have the situation of $\max c_i \max\{x_i, -x_i\}$, which cannot be easily modeled.

We consider two LP formulations for this problem.

1.
$$\begin{aligned} \min \ & \sum_{i=1}^{n} c_i z_i \\ \text{s.t.} \ & \left.\begin{aligned} z_i &\geqslant x_i \\ z_i &\geqslant -x_i \end{aligned}\right\} i = 1, \ldots, m \\ & A\bar{x} \geqslant \bar{b} \end{aligned}$$

z_i models $|x_i|$

two constraints imply $z_i \geqslant 0$

Notice that this approach mirrors the use of z previously to model $\min \max_{i=1,\ldots,m} \{\bar{c}_i^T \bar{x} + d_i\}$.

2. Using the idea of $x^{\pm}$ to model x urs:

$$\begin{aligned} \min \ & \sum_{i=1}^{n} c_i (x_i^+ + x_i^-) \\ \text{s.t.} \ & A(\bar{x}^+ - \bar{x}^-) \geqslant \bar{b} \\ & \bar{x}^+, \bar{x}^- \geqslant \bar{0} \end{aligned}$$

$$\bar{x}^+ = \begin{bmatrix} x_1^+ \\ x_2^+ \\ \vdots \\ x_n^+ \end{bmatrix} \qquad \bar{x}^- = \begin{bmatrix} x_1^- \\ \vdots \\ x_n^- \end{bmatrix}$$

x_i^+, x_i^-: model the positive/negative part of x_i — in the optimal solution, only one of them could be > 0. Both could be 0 (modeling $x_i = 0$).
$c_i \geqslant 0$ forces only one of x_i^+, x^- to be > 0, if at all.

Say $x_i = 3$. We could write $x_i = 0 - 3$ or $x_i = 2 - 5$ or even $x_i = 2023 - 2026$. The pairs (x_i^+, x_i^-) are $(0,3), (2,5)$, or $(2023, 2026)$. Each x_i^+ and x_i^- is $\geqslant 0$. Among all such pairs, we want to pick one with the smallest $x_i^+ + x_i^-$ sum. Indeed, we pick $(0,3)$! With $c_i \geqslant 0$, we get the same result when we want to minimize $c_i(x_i^+ + x_i^-)$.

Example: Write an LP formulation for this problem.

$$\begin{aligned} \min \quad & 2x_1 + |x_2| \\ \text{s.t.} \quad & 5x_1 + 3x_2 \geq 4 \\ & |x_1| + x_2 \leq 3 \\ & x_1, x_2 \text{ urs} \end{aligned}$$

We have $|x_2| = \max\{x_2, -x_2\}$, and

$$|x_1| + x_2 \leq 3$$

"equivalent to" $\equiv \max\{x_1, -x_1\} + x_2 \leq 3$

$$\equiv \max\{x_1 + x_2, -x_1 + x_2\} \leq 3$$

Here is the LP formulation, using the ideas we just introduced:

$$\begin{aligned} \min \quad & 2x_1 + z_2 \\ \text{s.t.} \quad & z_2 \geq x_2 \\ & z_2 \geq -x_2 \\ & 5x_1 + 3x_2 \geq 4 \\ & x_1 + x_2 \leq 3 \\ & -x_1 + x_2 \leq 3 \end{aligned}$$

Notice that z_2 models $|x_2|$ (the coefficient $c_2 = 1$ here).

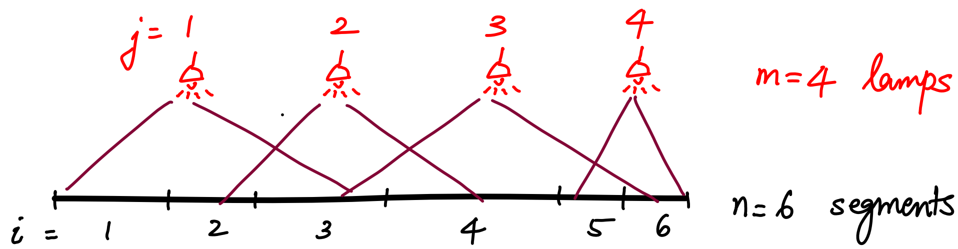
Since x_1, x_2 are urs to start with, we do not mention any sign restrictions here. Even though z_2 is not explicitly declared to be ≥ 0, the first two constraints ensure $z_2 \geq 0$.

2. Back to Lighting Problem (BT-ILO Prob 1.8):

d.V.s

Exercise 1.8 (Road lighting) Consider a road divided into n segments that is illuminated by m lamps. Let p_j be the power of the jth lamp. The illumination I_i (d.V.s) of the ith segment is assumed to be $\sum_{j=1}^{m} a_{ij} p_j$, where a_{ij} are known coefficients (data). Let I_i^* (data) be the desired illumination of road i.

We are interested in choosing the lamp powers p_j so that the illuminations I_i are close to the desired illuminations I_i^* (data). Provide a reasonable linear programming formulation of this problem. Note that the wording of the problem is loose and there is more than one possible formulation.



<u>Interpretation 2</u> Get <u>"as close as possible"</u> to I_i^* (could be above or below).

Minimize total deviations from I_i^*'s.

d.V.'s p_j, I_i, e_i ← The e_i's here model $|I_i - I_i^*|$, and not excess.

$$\min \sum_{i=1}^{n} e_i \qquad \text{(total deviation from } I_i^*)$$

s.t.

$$\left.\begin{array}{l} e_i \geq I_i - I_i^* \\ e_i \geq I_i^* - I_i \end{array}\right\} \text{(absolute value of } I_i - I_i^*)$$

$$I_i = \sum_{j=1}^{n} a_{ij} p_j, \quad i=1,\dots,n \quad \text{(illumination in segment } i)$$

$$p_j \geq 0 \ \forall j \quad \text{(non-neg)}$$

$e_i \geq 0$ is implied by these constraints.

BT-1LD **Exercise 1.10 (Production and inventory planning)** A company must deliver d_i units of its product at the end of the ith month. Material produced during a month can be delivered either at the end of the same month or can be stored as inventory and delivered at the end of a subsequent month; however, there is a storage cost of c_1 dollars per month for each unit of product held in inventory. The year begins with zero inventory. If the company produces x_i units in month i and x_{i+1} units in month $i+1$, it incurs a cost of $c_2|x_{i+1} - x_i|$ dollars, reflecting the cost of switching to a new production level. Formulate a linear programming problem whose objective is to minimize the total cost of the production and inventory schedule over a period of twelve months. Assume that inventory left at the end of the year has no value and does not incur any storage costs. assume ≥ 0!

d_i = demand at end of month i,

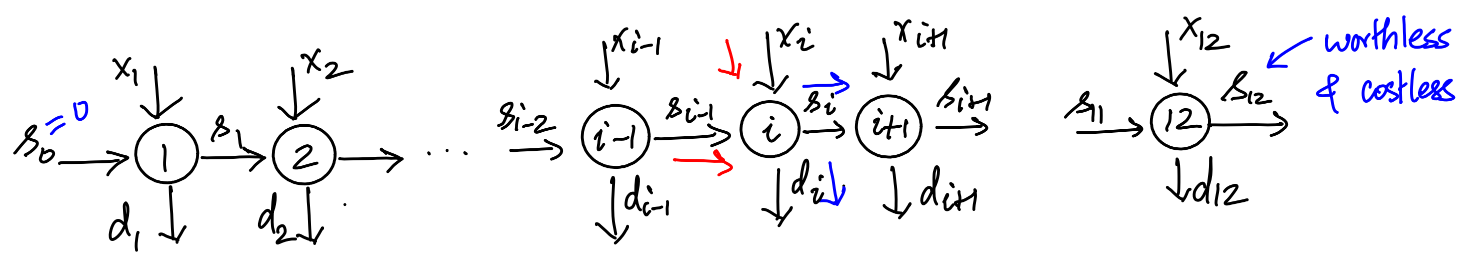
C_1 = storage cost,

x_i = # units produced in month i, $i = 0, 1, \ldots, 12$ (d.v.'s),

$C_2|x_{i+1} - x_i|$ = cost of switching production levels.

Let s_i = # units in inventory at end of month i; $i = 0, 1, \ldots, 12$ (d.v.'s).

Here is a visualization of the set up:



Node (i) represents the "flow" of units in month i.
We want "flow balance", i.e., total inflow = total outflow.

$$\underbrace{s_{i-1} + x_i}_{\text{inflow}} = \underbrace{s_i + d_i}_{\text{outflow}}, \quad i = 1, \ldots, 12.$$

Goal: minimize $\sum_{i=1}^{11} c_1 s_i + \sum_{i=0}^{11} c_2 \underbrace{|x_{i+1} - x_i|}_{z_i \text{ to model}}$

We need another set of d.v.s, say z_i, to model $|x_{i+1} - x_i|$. We use the same technique as used before to model absolute value terms in min-objective functions.

Here's the full LP:

$$\min \quad \sum_{i=1}^{11} c_1 s_i + \sum_{i=0}^{11} c_2 z_i$$

s.t.

$$\left.\begin{aligned} z_i &\geq x_{i+1} - x_i \\ z_i &\geq x_i - x_{i+1} \end{aligned}\right\} \text{ for } i = 0, \dots, 11 \quad \text{(absolute value of } x_{i+1} - x_i\text{)}$$

$$s_{i-1} + x_i = d_i + s_i, \quad i = 1, \dots, 12 \quad \text{(inventory balance)}$$

$$\left.\begin{aligned} x_0 &= 0 \\ s_0 &= 0 \end{aligned}\right\} \text{ (starting production/inventory)}$$

$$x_i, s_i \geq 0 \quad \text{(non-neg)} \qquad z_i \geq 0 \text{ automatically}$$

Notice we are able to write all constraints compactly by adding the extra variables s_0 and x_0.

How to solve LPs in 2D Graphically

Consider the following LP in 2D:

$$\begin{aligned} \min \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 2 \\ & 3x_1 + x_2 \geq 4 \\ & 3x_1 + 2x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

We demonstrate how we can "plot" this LP, and solve it in that process. Later on, we will extend these techniques to higher dimensions using linear algebra.

The set of all points (x_1, x_2) which satisfy all constraints (including nonnegativity) is called the **feasible region** of the LP.

We first plot the feasible region.

More in the next lecture...