MATH 230 - SOLUTIONS TO PRACTICE MIDTERM

1. The system can be rearranged to the following system.

$$\begin{aligned} x_1 + x_2 - 3x_3 + x_4 &= 6 \\ x_1 \quad\; - 2x_3 + x_4 &= 5 \\ -x_1 - x_2 + 3x_3 \quad\; &= -3 \end{aligned}$$

(a)

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ -2 \\ 3 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_4 = \begin{bmatrix} 6 \\ 5 \\ -3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & -3 & 1 \\ 1 & 0 & -2 & 1 \\ -1 & -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -3 \end{bmatrix}$$

(c)

$$\left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 6 \\ 1 & 0 & -2 & 1 & 5 \\ -1 & -1 & 3 & 0 & -3 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & -3 & 1 & 6 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[\substack{\text{and then} \\ R_2 \times (-1)}]{R_1 + R_2 - R_3} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

x_3 is free

$$\begin{aligned} x_1 - 2x_3 &= 2 \\ x_2 - x_3 &= 1 \\ x_4 &= 3 \end{aligned}$$

x_3 is free

The parametric vector form of the general solution is

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} s, \quad s \in \mathbb{R}.$$

(d) If the solutions to $A\bar{x} = \bar{b}$ are given as $\bar{x} = \bar{p} + \bar{q}s$, for $s \in \mathbb{R}$, the solutions to $A\bar{x} = \bar{0}$ are given as $\bar{x} = \bar{q}s$.

So, solutions to $A\bar{x} = \bar{0}$ are

$$\bar{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} s, \quad s \in \mathbb{R}.$$

2. The system $A\bar{x}=\bar{b}$ here consists of $A=[\bar{v}\ \bar{u}]$, and $\bar{b}=\bar{w}$. Since $3\bar{u}-4\bar{v}+\bar{w}=\bar{0}$, one solution to the given system is given by $\bar{x}=\begin{bmatrix}4\\-3\end{bmatrix}$, as $4\bar{v}-3\bar{u}=\bar{w}$.

3. We can start with $A=\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\end{bmatrix}$, which has a pivot in every row, and hence its columns span $\mathbb{R}^3$. Then we can use EROs to convert A to have every entry non-zero (the particular $\bar{b}$ given does not matter).

$$\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\end{bmatrix}\xrightarrow[R_2+R_3]{R_1+R_2+R_3}\begin{bmatrix}1&1&1&3\\0&1&1&2\\0&0&1&1\end{bmatrix}\xrightarrow[R_3+R_1]{R_2+R_1}\begin{bmatrix}1&1&1&3\\1&2&2&5\\1&1&2&4\end{bmatrix}$$

4. (a) We reduce $A=[\bar{v}_1\ \bar{v}_2\ \bar{v}_3\ \bar{v}_4]$ to echelon form.

$$A=\begin{bmatrix}4&2&-2&3\\1&6&5&0\\3&1&1&2\end{bmatrix}\xrightarrow[R_3-3R_2]{R_1-4R_2}\begin{bmatrix}0&-22&-22&3\\1&6&5&0\\0&-17&-14&2\end{bmatrix}\xrightarrow{R_1\times(-\frac{1}{22})}\begin{bmatrix}0&1&1&-3/22\\1&6&5&0\\0&-17&-14&2\end{bmatrix}$$

$$\xrightarrow{R_3+17R_1}\begin{bmatrix}0&1&1&-3/22\\1&6&5&0\\0&0&3&-7/22\end{bmatrix}\xrightarrow{R_1\rightleftharpoons R_2}\begin{bmatrix}1&6&5&0\\0&1&1&-3/22\\0&0&3&-7/22\end{bmatrix}$$

A has a pivot in every row, hence $\{\bar{v}_1,\bar{v}_2,\bar{v}_3,\bar{v}_4\}$ spans $\mathbb{R}^3$.

(b) No. $\bar{v}_1,\bar{v}_3\in\mathbb{R}^3$, and hence they cannot span $\mathbb{R}^2$.

5. $T(\bar{x}) = A\bar{x}$, where $A = [T(\bar{e}_1)\ T(\bar{e}_2)] = \begin{bmatrix} 2 & 3 \\ 1 & k \\ h & 0 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 3 \\ 1 & k \\ h & 0 \end{bmatrix} \xrightarrow[R_3 - hR_2]{R_1 - 2R_2} \begin{bmatrix} 0 & 3-2k \\ 1 & k \\ 0 & -hk \end{bmatrix} \xrightarrow{R_1 \rightleftarrows R_2} \begin{bmatrix} 1 & k \\ 0 & 3-2k \\ 0 & -hk \end{bmatrix}$$

(a) T is 1-to-1 if A has a pivot in every column. This condition holds true if $3-2k \neq 0$ or $-hk \neq 0$, i.e., if $k \neq \frac{3}{2}$, or when $h \neq 0, k \neq 0$. The set of values of h, k can be described as

$$k \in \mathbb{R}/\{3/2\},\ h \in \mathbb{R} \quad \text{or} \quad h \in \mathbb{R}/\{0\},\ k \in \mathbb{R}/\{0\}.$$

(b) T is onto if there is a pivot in every row, which is not possible here, as A can have at most 2 pivots.

6. $A = \begin{bmatrix} 1 & a \\ a & a+2 \end{bmatrix} \xrightarrow{R_2 - aR_1} \begin{bmatrix} 1 & a \\ 0 & a+2-a^2 \end{bmatrix} \neq 0$

The two vectors are LI if A has a pivot in every column. For this condition to hold true, we need $a+2-a^2 \neq 0$. We first solve $a^2 - a - 2 = 0$, which can be simplified to $(a-2)(a+1) = 0$, giving $a = 2, -1$ as the solutions. Hence the given vectors are LI for $a \in \mathbb{R}/\{2, -1\}$.

7. (a) We need a pivot in every row. The possible echelon forms are $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \end{bmatrix}$, $\begin{bmatrix} \blacksquare & * & * \\ 0 & 0 & \blacksquare \end{bmatrix}$, and $\begin{bmatrix} 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$.

(b) The only possible echelon form is $\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$.

8. (a) FALSE. Since $A\bar{x}=\bar{b}$ is not consistent for some $\bar{b}$, A does not have a pivot in every row. Since A is $n\times n$, A does not have a pivot in every column here. Hence $A\bar{x}=\bar{0}$ has free variable(s).

(b) FALSE. We get the solutions to $A\bar{x}=\bar{b}$ by adding some constant vector to the solutions of $A\bar{x}=\bar{0}$, and not necessarily $\bar{b}$.

(c) FALSE. $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$ has 1 vector with 2 entries, but is LD.

(d) FALSE. $x \longmapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} x$ is a 1-to-1 LT from $\mathbb{R}^1$ to $\mathbb{R}^2$ which is not on-to, as $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ does not have a pivot in every row.