Honors Linear Algebra (Math 230) – Spring 2011
Practice Midterm Exam

- There are **eight** problems in this exam.
- Points for each problem are given in parenthesis.
- Total points add to 110; you will be graded for 100 points.
- Show all work.
- Provide appropriate **justifications** where required.

1. **(20)** Consider the following system of linear equations.

$$\begin{aligned} x_2 + x_1 - 3x_3 + x_4 &= 6 \\ -2x_3 + x_4 + x_1 &= 5 \\ -x_1 - x_2 + 3x_3 &= -3 \end{aligned}$$

   (a) Write the system as a vector equation.

   (b) Write the system as a matrix equation $A\mathbf{x} = \mathbf{b}$.

   (c) Solve the system by reducing its augmented matrix to reduced row echelon form. Write the general solution to the system in the parametric-vector form.

   (d) From the solution to the system $A\mathbf{x} = \mathbf{b}$ in Part (c) above, write down the solutions to the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$. You must *not* solve the homogeneous system from scratch.

2. **(8)** Let

$$\mathbf{u} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \quad \text{and } \mathbf{w} = \begin{bmatrix} -4 \\ 17 \\ -13 \end{bmatrix}.$$

   It can shown that $3\mathbf{u} - 4\mathbf{v} + \mathbf{w} = \mathbf{0}$. Use this fact (and *no row operations*) to find a solution to the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 2 & 4 \\ 5 & 1 \\ -1 & 3 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and } \mathbf{b} = \begin{bmatrix} -4 \\ 17 \\ -13 \end{bmatrix}.$$

3. **(14)** Construct a 3×4 matrix A with every entry non-zero such that the following vector $\mathbf{b}$ is in the span of the columns of A. Justify your answer.

$$\mathbf{b} = \begin{bmatrix} -3 \\ 0 \\ 4 \end{bmatrix}$$

4. **(12)** Let

$$\mathbf{v}_1 = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

   (a) Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ span $\mathbb{R}^3$? Why or why not?

   (b) Does $\{\mathbf{v}_1, \mathbf{v}_2\}$ span $\mathbb{R}^2$? Why or why not?

5. **(14)** The images of the unit vectors in $\mathbb{R}^2$ under the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ are given as

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}, \quad \text{and } T(\mathbf{e}_2) = \begin{bmatrix} 3 \\ k \\ 0 \end{bmatrix}.$$

   Determine all the values of the parameters h and k for which the following statements are true.

   (a) T is one-to-one.

   (b) T maps $\mathbb{R}^2$ onto $\mathbb{R}^3$.

6. **(14)** Determine the values of a for which $\left\{ \begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} a \\ a+2 \end{bmatrix} \right\}$ is linearly independent.

7. **(12)** Describe all possible echelon forms of matrix A in each of the following cases.

   (a) A is a 2×3 matrix whose columns span $\mathbb{R}^2$.

   (b) A is a 3×3 matrix whose columns span $\mathbb{R}^3$.

8. **(16)** Decide whether each of the following statements is *True* or *False*. Justify your answer.

   (a) Let A be an $n \times n$ matrix. If the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$ in $\mathbb{R}^n$, then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

   (b) The solutions of $A\mathbf{x} = \mathbf{b}$ are obtained by adding the vector $\mathbf{b}$ to the solutions of $A\mathbf{x} = \mathbf{0}$.

   (c) If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.

   (d) If a linear transformation is one-to-one, then it must also be on-to.