MATH 230 - Brief Solutions To Practice Final

1.

$$A=\begin{bmatrix}1&2&4&1\\0&3&9&6\\0&0&0&2\\1&2&4&2\end{bmatrix}\xrightarrow{R_4-R_1}\begin{bmatrix}1&2&4&1\\0&3&9&6\\0&0&0&2\\0&0&0&1\end{bmatrix}\xrightarrow[R_3\rightleftharpoons R_4]{R_2\times\frac{1}{3}}\begin{bmatrix}1&2&4&1\\0&1&3&2\\0&0&0&1\\0&0&0&2\end{bmatrix}\xrightarrow[R_4-2R_3]{R_1-2R_2}\begin{bmatrix}1&0&-2&-3\\0&1&3&2\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow[R_2-2R_3]{R_1+3R_3}\begin{bmatrix}1&0&-2&0\\0&1&3&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

Solutions to $A\bar{x}=\bar{0}$ are:

$x_1 = 2x_3$

$x_2 = -3x_3$, x_3 free

$x_4 = 0$

(a) A basis for Col A is $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}2\\3\\0\\2\end{bmatrix},\begin{bmatrix}1\\6\\2\\2\end{bmatrix}\right\}$.

(pivot columns from the original A matrix)

(b) Solutions to $A\bar{x}=\bar{0}$ can be written as

$$\bar{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}2\\-3\\1\\0\end{bmatrix}x_3,\quad x_3\in\mathbb{R}.$$

Hence, $\left\{\begin{bmatrix}2\\-3\\1\\0\end{bmatrix}\right\}$ is a basis for Nul A.

(c) dim Nul A = 1. (# free variables)

(d) rank A = 3. (# pivot columns)

(e) det A = 0. ($A\bar{x}=\bar{0}$ has non-trivial solutions, so A is not invertible, i.e., det A = 0.)

2. It is evident that $A \in \mathbb{R}^{2\times 2}$. The two mappings can be written together as

$$A\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix} = \begin{bmatrix}1 & 3\\ 1 & 1\end{bmatrix}.$$

Notice that the matrix $\begin{bmatrix}1 & 2\\ 3 & 7\end{bmatrix}$ is invertible. It's inverse is $\frac{1}{1}\begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix} = \begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix}$. Multiplying the above equation on the right by this inverse, we get

$$A = \begin{bmatrix}1 & 3\\ 1 & 1\end{bmatrix}\begin{bmatrix}7 & -2\\ -3 & 1\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 4 & -1\end{bmatrix}.$$

3. $A = \begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 2\end{bmatrix}$ will have $\text{rank}(A) = 2$, as the first two columns are same, and the third column is not a scalar multiple of this column. Hence there will be two pivots.
For $\bar{b} = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$, $A\bar{b} = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix} \neq \bar{0}$, showing $\bar{b} \notin \text{Nul}\, A$.

4. A is similar to $B \Rightarrow \exists$ invertible matrix P such that $B = P^{-1}AP$. —(1)
B is similar to $C \Rightarrow \exists$ invertible matrix Q such that $C = Q^{-1}BQ$. —(2)
Plugging in the expression for B from (1) into (2), we get

$$C = \underbrace{Q^{-1}(P^{-1}}AP\underbrace{)Q} = (Q^{-1}P^{-1})A(PQ)$$ (distributive property of matrix multiplication)

$$= R^{-1}AR, \text{ where } R = PQ. \quad ((AB)^{-1} = B^{-1}A^{-1})$$

Since both P and Q are invertible, so is $R = PQ$. Thus, $C = R^{-1}AR$, showing that C is similar to A.

5. $A - \lambda I = \begin{bmatrix} 3-3 & 0 & 1 \\ 6 & 5-3 & 7 \\ -3 & -1 & -2-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 6 & 2 & 7 \\ -3 & -1 & -5 \end{bmatrix} \xrightarrow{R_2+2R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -3 \\ -3 & -1 & -5 \end{bmatrix} \xrightarrow[R_3+5R_1]{R_2+3R_1}$

$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -3 & -1 & 0 \end{bmatrix} \xrightarrow[\text{EROs}]{\text{3 more}} \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left\} \begin{array}{l} x_1 = -\frac{1}{3}x_2 \\ x_2 \text{ free} \\ x_3 = 0 \end{array} \right. ; \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix} x_2, \; x_2 \in \mathbb{R}.$

So, $\lambda = 3$ is an eigenvalue, and $\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$ is a corresponding eigenvector.

6. Notice that the highest degree of the five polynomials given is 2, and hence $H = \text{span}(p_1(t), p_2(t), p_3(t), p_4(t), p_5(t))$ will have $\dim H = 3$ $(2+1)$. To find bases for H, we write the corresponding coefficient vectors ($\bar{v}_j$ for $p_j(t)$, $j = 1, \dots, 5$) such that if $p_j(t) = a_0 + a_1 t + a_2 t^2$, then $\bar{v}_j = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$. Hence we get $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 0 \end{bmatrix}$, $\bar{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\bar{v}_5 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

Reordering, we write $A = [\bar{v}_3 \; \bar{v}_1 \; \bar{v}_2 \; \bar{v}_4 \; \bar{v}_5] = \begin{bmatrix} -4 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$.

so as to spot LI columns easily.

We see that $\{\bar{v}_3, \bar{v}_1, \bar{v}_4\}$ and $\{\bar{v}_3, \bar{v}_2, \bar{v}_5\}$ are two choices for a basis for Col A. Hence $\{p_1(t), p_3(t), p_4(t)\}$ and $\{p_2(t), p_3(t), p_5(t)\}$ are two different bases for H.

Note: Ideally, we should have considered $\bar{v}_j$'s as 4-vectors, since we are talking about $\mathbb{P}_3$. But we reach the same conclusion(s) here.

7. $A\left(A^{-1}(A+X)B = C\right)$

$\Rightarrow \left(\underbrace{(AA^{-1})}_{I}(A+X)B = AC\right)B^{-1}$ $\quad (A(BC) = (AB)C)$

$\Rightarrow ((A+X)\underbrace{B)B^{-1}} = ACB^{-1}$ $\quad (AA^{-1} = I)$

$\Rightarrow A+X = ACB^{-1}$ $\quad (BB^{-1} = I)$

$\Rightarrow X = ACB^{-1} - A$ $\quad$ (matrix addition)

$\Rightarrow \boxed{X = A(CB^{-1} - I)}$ $\quad (AB+AC = A(B+C))$

8. $$\begin{bmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}}_{\bar{v}_1} a + \underbrace{\begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}}_{\bar{v}_2} b + \underbrace{\begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix}}_{\bar{v}_3} c, \quad a,b,c \in \mathbb{R}.$$

Hence the set of vectors in question is $\operatorname{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$. But, notice that $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is **not** LI. In fact, one could check that $\bar{v}_1 - 2\bar{v}_2 = \bar{v}_3$, for instance. But we do have $\bar{v}_i \neq c\bar{v}_j$ for all $i,j = 1,2,3$, $i \neq j$. Hence we could choose any pair of the vectors to get a basis, e.g., $\{\bar{v}_1, \bar{v}_2\}$ or $\{\bar{v}_2, \bar{v}_3\}$.

9.

(a) $T(\bar{x}) = A\bar{x} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 4 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 12 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 6\bar{x}.$

(b) $\lambda = 6$ is an eigenvalue of A, and $\bar{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ is a corresponding eigenvector.

10.

With $\lambda = 2$, $A - \lambda I = \begin{bmatrix} 3-2 & 5 & 6 \\ 0 & 2-2 & h \\ 0 & 0 & 2-2 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 6 \\ 0 & 0 & h \\ 0 & 0 & 0 \end{bmatrix}$

To get two linearly independent eigenvectors corresponding to $\lambda = 2$, we need two free variables in the system $(A - \lambda I)\bar{x} = \bar{0}$. To have x_3 free, we must have $\boxed{h = 0.}$.

11.

$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -2 & 3 \\ 0 & 1-\lambda & 0 \\ 6 & 7 & -2-\lambda \end{vmatrix} = (-1)^{2+2}(1-\lambda)\begin{vmatrix} 5-\lambda & 3 \\ 6 & -2-\lambda \end{vmatrix}$

expand along Row 2

$= (1-\lambda)\left[-(5-\lambda)(2+\lambda) - 18\right]$

$= (1-\lambda)\left[\lambda^2 - 3\lambda - 10 - 18\right] = (1-\lambda)(\lambda^2 - 3\lambda - 28) = -(\lambda-1)(\lambda+4)(\lambda-7)$

$= \mathbf{-\lambda^3 + 4\lambda^2 + 25\lambda - 28}$, which is the characteristic polynomial.

To find the eigenvalues, we solve $\det(A - \lambda I) = 0$, i.e., $-(\lambda-1)(\lambda+4)(\lambda-7) = 0$, which gives $\lambda = 1, -4, 7$ as the eigenvalues.

12. TRUE/FALSE

(a) FALSE. $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ have the same reduced echelon form (B itself). But $\text{Col}\, A = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ and $\text{Col}\, B = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$.

(b) TRUE. Columns of B are LI $\Rightarrow$ no free variables
$\Rightarrow B\bar{x} = \bar{0}$ has only trivial solution.

(c) TRUE. If AB^{-1} is invertible, then so are both A and B^{-1} (by IMT). Since B^{-1} is invertible, so is $(B^{-1})^{-1} = B$; and since A and B are invertible, $(AB)^{-1}$ exists, $(AB)^{-1} = B^{-1}A^{-1}$.

(d) FALSE. True only if the plane passes through the origin.

(e) TRUE. $\underbrace{A^{-1}(A}\bar{x} = \lambda \bar{x}) \Rightarrow \left(I\bar{x} = \lambda A^{-1}\bar{x}\right)\frac{1}{\lambda} \Rightarrow A^{-1}\bar{x} = \left(\frac{1}{\lambda}\right)\bar{x}$.