Honors Linear Algebra (Math 230) – Spring 2011
Practice Final Exam

- There are **twelve** problems in this exam.
- Points (given in parenthesis) add to 105. You will be graded for 100.
- Show all work.
- Provide appropriate **justifications** where required.
- Vectors are denoted by **bold**, lower case letters, e.g., $\mathbf{x} \in \mathbb{R}^n, \mathbf{y}, \mathbf{0}$, etc. Matrices are denoted by capital letters, e.g., A, B, etc.

1. **(10)**

$$\text{Let} \quad A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 3 & 9 & 6 \\ 0 & 0 & 0 & 2 \\ 1 & 2 & 4 & 2 \end{bmatrix}.$$

   (a) Determine a basis for $\operatorname{Col} A$.

   (b) Determine a basis for $\operatorname{Nul} A$.

   (c) What is $\dim \operatorname{Nul} A$? Explain.

   (d) What is $\operatorname{rank} A$? Explain.

   (e) What is $\det A$?

2. **(8)** Find the matrix A such that the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ maps $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

3. **(8)** Construct a 3×3 matrix A with every entry *non-zero* and rank 2, and a vector $\mathbf{b}$ that is *not* in $\operatorname{Nul} A$.

4. **(10)** Let A, B, C be $n \times n$ matrices. Show that if A is similar to B and B is similar to C, then A is similar to C.

5. **(7)** Is $\lambda = 3$ an eigenvalue of $A = \begin{bmatrix} 3 & 0 & 1 \\ 6 & 5 & 7 \\ -3 & -1 & -2 \end{bmatrix}$? If yes, find one corresponding eigenvector.

6. **(7)** Let H be the subspace of $\mathbb{P}_3$ spanned by the polynomials $p_1(t) = 1 + t,\ p_2(t) = 1 - t,\ p_3(t) = -4,\ p_4(t) = t + t^2,\ p_5(t) = 1 + 2t - 2t^2$. Find *two* different bases for H.

7. **(8)** Let A, B, and C be $n \times n$ invertible matrices. Solve the following equation for X. Justify each step in your solution.

$$A^{-1}(A + X)B = C.$$

8. **(10)** Find a basis for the set of vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} a - 2b + 5c \\ 2a + 5b - 8c \\ -a - 4b + 7c \\ 3a + b + c \end{bmatrix}$, where $a, b, c \in \mathbb{R}$.

9. **(6)** Define the linear transformation $T(\boldsymbol{x}) = A\boldsymbol{x}$ with $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 4 \\ -1 & 5 & 2 \end{bmatrix}$.

   (a) Find the image of $\boldsymbol{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ under this transformation.

   (b) Based on the result obtained above (and without doing any further calculations), find an eigenvalue and a corresponding eigenvector of A.

10. **(8)** Let $A = \begin{bmatrix} 3 & 5 & 6 \\ 0 & 2 & h \\ 0 & 0 & 2 \end{bmatrix}$. What should be the value of h so that there are two linearly independent eigenvectors of A corresponding to the eigenvalue $\lambda = 2$.

11. **(8)** Find the characteristic polynomial and the eigenvalues of A where

$$A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}.$$

12. **(15)** Decide whether each of the following statements is *True* or *False*. Justify your answer.

   (a) If matrices A and B have the same reduced row echelon form, then $\operatorname{Col} A = \operatorname{Col} B$.

   (b) If B is a 5×4 matrix with linearly independent columns, then $\operatorname{Nul} B = \{\mathbf{0}\}$.

   (c) If AB^{-1} is invertible, so is AB.

   (d) A plane in $\mathbb{R}^3$ is a 2-dimensional subspace of $\mathbb{R}^3$.

   (e) If λ is a nonzero eigenvalue of an invertible matrix A, then $1/\lambda$ is an eigenvalue of A^{-1}.