Honors Introductory Linear Algebra (Math 230) – Spring 2011
Midterm Examination

- There are **eight** problems and **six** pages in this exam.
- Provide appropriate **justifications** where required.
- Total points given in parentheses add to 105. You will be graded for 100 points.

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1. **(12)** Find the matrix of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ that first projects points in $\mathbb{R}^4$ to $\mathbb{R}^2$ by ignoring the 3rd and 4th coordinates, then horizontally shears the resulting vectors such that $\mathbf{e}_2 \in \mathbb{R}^2$ is mapped to $\mathbf{e}_2 + 2\mathbf{e}_1$, while leaving $\mathbf{e}_1$ unchanged, and then reflects the vectors thus obtained through the vertical axis.

2. **(19)** Consider the following system of linear equations.

$$\begin{aligned} x_3 + x_4 + x_1 &= 1 \\ -4x_4 + x_2 &= 0 \\ -2x_1 + 2x_2 - x_3 - x_4 &= 3 \end{aligned}$$

   (a) Write the system as a vector equation.

   (b) Write the system as a matrix equation $A\mathbf{x} = \mathbf{b}$.

   (c) Solve the system by reducing its augmented matrix to reduced row echelon form. Write the general solution to the system in the parametric-vector form.

   (d) From the solution to the system $A\mathbf{x} = \mathbf{b}$ in Part (c) above, write down the solutions to the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$. You must *not* solve the homogeneous system from scratch.

3. **(12)** Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}, \quad \text{and } \mathbf{w} = \begin{bmatrix} 0.5 \\ 2 \\ -5 \end{bmatrix}.$$

   It can shown that $3\mathbf{u} - \mathbf{v} = 2\mathbf{w}$. Use this fact (and *no row operations*) to find a non-trivial solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$, where

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 9 & -4 & 5 \\ -3 & 10 & 7 \end{bmatrix}.$$

4. **(10)** Construct a 3×3 matrix A with every entry non-zero such that the following vector $\mathbf{b}$ is ***not*** in the span of the columns of A. Justify your answer.

$$\mathbf{b} = \begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix}$$

5. **(14)** Let

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_4 = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}.$$

   (a) Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ span $\mathbb{R}^3$? Why or why not?

   (b) Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ span $\mathbb{R}^3$? Why or why not?

6. **(14)** Determine the values of a for which the LT defined by $\mathbf{x} \mapsto A\mathbf{x}$ is *onto*, when $A = \begin{bmatrix} 2 & 2a \\ a & 8-2a \end{bmatrix}$.

7. **(12)** Consider the following system.

$$\begin{aligned} x_1 + 3x_2 &= k \\ kx_1 + hx_2 &= 2 \end{aligned}$$

   Determine all the values of the parameters h and k for which each of the following statements are true.

   (a) The system has no solution.

   (b) The system has a unique solution.

   (c) The system has many solutions.

8. **(12)** Decide whether each of the following statements is *True* or *False*. Justify your answer.

   (a) A 3×3 matrix can have at most three echelon forms.

   (b) If $A \in \mathbb{R}^{m \times n}$ has a pivot in every column, then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^m$.

   (c) If $A\mathbf{x} = \mathbf{b}$ has many solutions, then $A\mathbf{x} = \mathbf{c}$ also has many solutions.

   (d) If A is an $m \times n$ matrix, the range of the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $\mathbb{R}^m$.