Honors Introductory Linear Algebra (Math 230) – Spring 2011
Final Examination

- There are **twelve** problems and **eight** pages in this exam.
- Points (in parentheses) add to 105. You will be graded for 100 points.
- Provide appropriate **justifications** where required.
- Good luck!

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1. **(8)** Find the characteristic polynomial and the eigenvalues of A where

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

2. **(8)**

$$\text{Let} \quad A = \begin{bmatrix} 1 & 3 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & -1 & 2 \\ -2 & -6 & -2 & 6 & 0 \end{bmatrix}.$$

   (a) Determine a basis for $\operatorname{Col} A$.

   (b) Determine a basis for $\operatorname{Nul} A$.

   (c) What is $\dim \operatorname{Nul} A$? Explain.

   (d) What is $\operatorname{rank} A$? Explain.

3. **(12)** Let A and B be invertible $n \times n$ matrices, and let $\mathbf{x}$ be an eigenvector of the product AB corresponding to the nonzero eigenvalue λ. Show that the vector $B\mathbf{x}$ is an eigenvector of $(BA)^{-1}$. What is the corresponding eigenvalue? Justify your steps.

4. **(10)** Let $A + B$ and C be $n \times n$ invertible matrices. Solve the following equation for X. Justify each step in your solution.

$$C^{-1}(XB + XA)C = C^T.$$

5. **(9)** Let $A = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 4 & 11 \\ 0 & -1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 \\ 1 & 5 \\ 6 & -1 \end{bmatrix}$. Find $A^{-1}B$ *without* computing A^{-1}.

6. **(7)** Construct a 3×3 matrix A with rank 2, and list two different nonzero (i.e., with at least one entry nonzero) vectors in $\operatorname{Nul} A$. Justify.

7. **(8)** Let H be the subspace of $\mathbb{P}_3$ spanned by the polynomials $p_1(t) = t - t^2,\ p_2(t) = 1 + 2t + 4t^2,\ p_3(t) = -1 + t - 7t^2$. Find *two* different bases for H.

8. **(6)** Let $\det A = 3$ and $\det B = 2$. Evaluate each of the following quantities, if possible. Justify your answers.

   (a) $\det(2AB^T)$

   (b) $\det A^{-1} / \det B^{-1}$

   (c) $\det(A + B)$

9. **(8)** Let $A = \begin{bmatrix} 2 & 5 \\ k & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & h \end{bmatrix}$. What value(s) of h and k, if any, will make $AB = BA$?

10. **(6)** It is known that $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector of a 3×3 matrix A corresponding to the eigenvalue $\lambda = 0$. Is the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ one-to-one? Is it onto? Justify your answers.

11. **(8)** Let $A = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$.

    (a) Is $\lambda = 1$ and eigenvalue of A? If yes, find an associated eigenvector.

    (b) Is $\lambda = -2$ and eigenvalue of A? If yes, find an associated eigenvector.

    (c) Is $\mathbf{x} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ an eigenvector of A? If yes, find the corresponding eigenvalue.

    (d) Is $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ an eigenvector of A? If yes, find the corresponding eigenvalue.

12. **(15)** Decide whether each of the following statements is *True* or *False*. Justify your answer.

    (a) If $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$, then $\lambda = 0$ is an eigenvalue of A.

    (b) It could happen that $\det(A + B) = \det A + \det B$.

    (c) If $\mathbf{x}$ is an eigenvector of both matrices A and B, then it is also an eigenvector of AB.

    (d) $\det(-A) = -\det(A)$.

    (e) If the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, then the 3×3 matrix A is invertible.