MATH230 - SOLUTIONS TO FINAL EXAM

1. $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 3 & 5 \\ 4 & 1-\lambda & 6 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = (3-\lambda)\left[(2-\lambda)(1-\lambda)-12\right]$

$$= (3-\lambda)\left[\lambda^2-3\lambda-10\right] = (3-\lambda)(\lambda-5)(\lambda+2)$$

$$= -\lambda^3+6\lambda^2+\lambda-30 \leftarrow \text{characteristic polynomial}$$

The eigenvalues are the roots of $\det(A-\lambda I)=0$, i.e., $\lambda = 3, 5, -2$.

2. $\begin{bmatrix} 1 & 3 & 1 & -3 & 0 \\ 0 & 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & -1 & 2 \\ -2 & -6 & -2 & 6 & 0 \end{bmatrix} \xrightarrow[\text{then } R_1-R_2]{R_4+2R_1} \begin{bmatrix} 1 & 3 & 0 & 0 & -4 \\ 0 & 0 & 1 & -3 & 4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{then } R_3\times(-1)]{R_2-3R_3} \begin{bmatrix} 1 & 3 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

pivot columns

(a) Basis for Col A $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ -3 \\ -1 \\ 6 \end{bmatrix} \right\}$. (pivot columns)

(b) Solutions to $A\bar{x}=\bar{0}$ are given by $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 4 \\ 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} x_5$, $x_2, x_5 \in \mathbb{R}$.

$\Rightarrow$ A basis for Nul A is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(c) dim Nul A = 2 (# free variables)

(d) rank A = 3 (# pivot columns)

3. We have $AB\bar{x} = \lambda\bar{x}$ (definition of eigenvalue)

$$\Rightarrow \quad A^{-1}(AB\bar{x}) = \underbrace{A^{-1}A}_{I}B\bar{x} = \lambda A^{-1}\bar{x} \qquad (A^{-1} \text{ exists, } A(BC) = (AB)C).$$

$$\Rightarrow \quad \frac{1}{\lambda}B\bar{x} = A^{-1}\bar{x} = A^{-1}\underbrace{B^{-1}B}_{I}\bar{x} \qquad (\lambda \neq 0,\ B^{-1} \text{ exists})$$

$$\Rightarrow \quad (BA)^{-1}B\bar{x} = \frac{1}{\lambda}B\bar{x} \qquad \left((AB)^{-1} = B^{-1}A^{-1}\right)$$

Since B is invertible $B\bar{x} = \bar{0}$ has only the trivial solution. Since $\bar{x} \neq \bar{0}$ as it is an eigenvector, we get that $B\bar{x} \neq \bar{0}$ as well. Hence $B\bar{x}$ is an eigenvector of $(BA)^{-1}$ for the eigenvalue $\frac{1}{\lambda}$.

4.
$$\underbrace{C\,C^{-1}}_{I}(XB + XA)\underbrace{C\,C^{-1}}_{I} = CC^{T}C^{-1} \qquad (C \text{ is invertible})$$

$$\Rightarrow \quad IX(B+A)I = X(B+A) = CC^{T}C^{-1} \qquad \left(A(BC) = (AB)C,\ CC^{-1} = I\right)$$
$$\left(A(B+C) = AB + AC\right)$$

$$\Rightarrow \quad X\underbrace{(B+A)(A+B)^{-1}}_{I} = CC^{T}C^{-1}(A+B)^{-1} \qquad (B+A = A+B,\ A+B \text{ is invertible})$$

$$\Rightarrow \quad \boxed{X = CC^{T}C^{-1}(A+B)^{-1}}$$

5. With A invertible, $A^{-1}B$ is the solution to the system $AX = B$. Hence we can find $A^{-1}B$ by row reducing the augmented matrix $[A|B]$ to $[I|A^{-1}B]$.

$$\left[\begin{array}{ccc|cc} 1 & 3 & 8 & -3 & 5 \\ 2 & 4 & 11 & 1 & 5 \\ 0 & -1 & -3 & 6 & -1 \end{array}\right] \xrightarrow[R_3 \times -1]{R_2 - 2R_1} \left[\begin{array}{ccc|cc} 1 & 3 & 8 & -3 & 5 \\ 0 & -2 & -5 & 7 & -5 \\ 0 & 1 & 3 & -6 & 1 \end{array}\right] \xrightarrow[\text{then } R_2 \rightleftarrows R_3]{R_1 - 3R_3,\ R_2 + 2R_3} \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 15 & 2 \\ 0 & 1 & 3 & -6 & 1 \\ 0 & 0 & 1 & -5 & -3 \end{array}\right]$$

$$\xrightarrow[R_2 - 3R_3]{R_1 + R_3} \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 10 & -1 \\ 0 & 1 & 0 & 9 & 10 \\ 0 & 0 & 1 & -5 & -3 \end{array}\right] \qquad \text{Hence } A^{-1}B = \begin{bmatrix} 10 & -1 \\ 9 & 10 \\ -5 & -3 \end{bmatrix}.$$

6. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has $\text{rank}(A) = 2$, as it has two pivot columns.

$\bar{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$ give $A\bar{x}_1 = A\bar{x}_2 = \bar{0}$, thus $\bar{x}_1, \bar{x}_2 \in \text{Nul } A$.

7. The corresponding vectors are $\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} -1 \\ 1 \\ -7 \end{bmatrix}$.

$$A = [\bar{v}_1\ \bar{v}_2\ \bar{v}_3] = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 4 & -7 \end{bmatrix} \xrightarrow{R_3 + R_2 - 6R_1} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ has only two pivots.

Hence $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is LD. But $\bar{v}_i \neq c\bar{v}_j$ for $i,j = 1,2,3$, $i \neq j$ (evident from the negative sign entries). Hence $\{\bar{v}_1, \bar{v}_2\}$, $\{\bar{v}_2, \bar{v}_3\}$, and $\{\bar{v}_1, \bar{v}_3\}$ are all bases for $\text{span}(\bar{v}_1, \bar{v}_2, \bar{v}_3)$.

Thus, $\{p_1(t), p_2(t)\}$ and $\{p_1(t), p_3(t)\}$ are two different bases for H.

8. (a) $\det(2AB^T) = 2^n \det(A)\cdot\det(B)$ where $A, B \in \mathbb{R}^{n\times n}$.

Notice that we must have A and B of the same size for the product AB^T to be defined. Also, $\det(AB) = \det A \cdot \det(B)$, and $\det(B^T) = \det B$.

Hence we cannot evaluate the given expression without knowing the value of n.

(b) $$\det A^{-1} / \det B^{-1} = \left(\frac{1}{\det A}\right) \Big/ \left(\frac{1}{\det B}\right) = \frac{1/3}{1/2} = \frac{2}{3}$$

$\left(\det A^{-1} = \dfrac{1}{\det A} \text{ when } \det A \neq 0\right)$.

(c) $\det(A+B)$ cannot be evaluated. In particular, $\det(A+B) \neq \det A + \det B$.

9. $$AB = \begin{bmatrix} 2 & 5 \\ k & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & h \end{bmatrix} = \begin{bmatrix} 23 & 5h-10 \\ 4k+3 & h-5k \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & -5 \\ 3 & h \end{bmatrix}\begin{bmatrix} 2 & 5 \\ k & 1 \end{bmatrix} = \begin{bmatrix} 8-5k & 15 \\ 6+hk & h+15 \end{bmatrix}$$

$AB = BA$ gives $8-5k = 23 \Rightarrow k = \frac{-15}{5} = -3$

and $5h - 10 = 15 \Rightarrow h = \frac{25}{5} = 5$.

We also need $4k+3 = 6+hk$ and $h-5k = h+15$, which both hold for $h=5$, $k=-3$.

10. $\bar{x} \neq \bar{0}$ is an eigenvector for eigenvalue $\lambda = 0$. Hence $A\bar{x} = \lambda\bar{x} = \bar{0}$. As such $A\bar{x} = \bar{0}$ has a non-trivial solution. Hence A is not invertible, i.e., does not have a pivot in every row or in every column. Hence $T(\bar{x}) = A\bar{x}$ is neither onto nor one-to-one.

11. (a) $|A - \lambda I| = \begin{vmatrix} 1 & -4 \\ -1 & -2 \end{vmatrix} = -6 \neq 0$. Hence $\lambda = 1$ is not an eigenvalue.

(b) $A - \lambda I = \begin{bmatrix} 4 & -4 \\ -1 & 1 \end{bmatrix} \xrightarrow[\substack{R_2 \times (-1) \\ \text{then } R_1 \rightleftarrows R_2}]{R_1 + 4R_2} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(A - \lambda I) = 0$

$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2,\ x_2 \in \mathbb{R}$. Hence $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = -2$.

(c) $A\bar{x} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} -4 \\ 1 \end{bmatrix}$.

So $\bar{x} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue $\lambda = 3$.

(d) $A\bar{x} = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \neq \lambda\bar{x}$ for any λ.

Hence $\bar{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is not an eigenvector of A.

12. TRUE/FALSE.

(a) TRUE. Since $A\bar{x} = \bar{b}$ is inconsistent, A does not have a pivot in every row, and hence in every column (A is $n \times n$ here). Hence A has free variable(s), and so $A\bar{x} = \bar{0}$ has non-trivial solutions. Thus $\lambda = 0$ is an eigenvalue, as for any non-trivial solution $\bar{x}$ of $A\bar{x} = \bar{0}$, we have $A\bar{x} = \lambda \bar{x} = 0\bar{x} = \bar{0}$.

(b) TRUE. Just take $A = B = [0]$, for example.

(c) TRUE. Let $A\bar{x} = \lambda\bar{x}$ and $B\bar{x} = \mu\bar{x}$. Multiplying the second equation on the left by A, we get $AB\bar{x} = \mu A\bar{x} = \mu\lambda\bar{x}$. Hence $\bar{x}$ is an eigenvector of AB corresponding to eigenvalue $\mu\lambda$. Similarly, it is an eigenvector of BA as well.

(d) FALSE. $\det(-A) = (-1)^n \cdot \det(A)$.

(e) TRUE. Since A is 3x3, it has no free variables, and hence a pivot in every column. Thus A is invertible.