MATH 220 - Lecture 9 (09/17/2013)

Review

We are going to slow down a bit, as the other sections are a bit behind. We will review some concepts in this lecture.

$$A = \begin{array}{cc} & x_2 \\ \left[\begin{matrix} 1 & 2 \end{matrix}\right] \end{array},\quad x \text{ basic},\ x_2 \text{ free}$$

trivial solution to $A\bar{x} = 0$ is $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$[1\ 2 \mid 0]$ in reduced echelon form

$x_1 + 2x_2 = 0 \quad x_2 \text{ free} \qquad x_1 = -2s,\ s \in \mathbb{R}$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} s,\ s \in \mathbb{R}.$$

e.g. $s = 2,\ \bar{x} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} \qquad (-4) + 2(2) = 0$

So, **pivot in every column of A means $A\bar{x} = \bar{0}$ has only the trivial solution.**

Section 1.4 Page 41, Prob 18

$$B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix}.$$

Do the columns of B span $\mathbb{R}^4$?

Every vector in $\mathbb{R}^4$ can be written as a linear combination of the columns of B if B has a pivot in every row.

Equivalently, if B has a pivot in every row, the span of its columns is (all of) $\mathbb{R}^4$.

$$B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 2 & 9 & 5 & -7 \end{bmatrix} \xrightarrow{R_4 - 2R_1} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 2 & 6 & 7 \\ 0 & 1 & 3 & -11 \end{bmatrix} \xrightarrow[R_4 - R_2]{R_3 - 2R_2} \begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & -7 \end{bmatrix} \xrightarrow{R_4 + \frac{7}{15}R_3}$$

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since every row does not have a pivot, span {columns of B} $\neq \mathbb{R}^4$.

B has 3 pivots.

Q: Do columns of B span $\mathbb{R}^3$?

Reword: Can you write every vector in $\mathbb{R}^3$ as a combination of columns of B?

No! As columns of B sit in $\mathbb{R}^4$, and not in $\mathbb{R}^3$.

Every column of B has four entries, while any vector $\bar{u}$ in $\mathbb{R}^3$ has three entries. So, we cannot write $\bar{u}$ as $\bar{b}_1x_1 + \bar{b}_2x_2 + \bar{b}_3x_3 + \bar{b}_4x_4$.

Similarly, if B had only 2 pivots, its columns would still not span $\mathbb{R}^2$.

Prob 24, pg 41 (T/F)

24. a. Every matrix equation $A\mathbf{x} = \mathbf{b}$ corresponds to a vector equation with the same solution set.

b. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then $\mathbf{b}$ is in the set spanned by the columns of A.

c. Any linear combination of vectors can always be written in the form $A\mathbf{x}$ for a suitable matrix A and vector $\mathbf{x}$.

d. If the coefficient matrix A has a pivot position in every row, then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent.

e. The solution set of a linear system whose augmented matrix is $[\,\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \;\; \mathbf{b}\,]$ is the same as the solution set of $A\mathbf{x} = \mathbf{b}$, if $A = [\,\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3\,]$.

f. If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^m$.

(a) T. If $A = [\bar{a}_1 \, \bar{a}_2 \cdots \bar{a}_n]$, then $A\bar{x} = \bar{b}$ corresponds to the vector equation $\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots + \bar{a}_n x_n = \bar{b}$.

(b) T. If $\bar{x}$ is a solution, we can write $\bar{b} = \bar{a}_1 x_1 + \cdots + \bar{a}_n x_n$, where $\bar{a}_i$s are the columns of A.

(c) T. Same reason as above.

(d) F. Pivot in every row means $A\bar{x} = \bar{b}$ is consistent for every $\bar{b}$.

(e) T. From the definition.

(f) F. If columns of A span $\mathbb{R}^m$, then $A\bar{x} = \bar{b}$ has a solution for every $\bar{b} \in \mathbb{R}^m$!

Section 1.5 pg 48, prob 26.

A is the 3×3 zero matrix. Describe solutions of $A\bar{x}=\bar{0}$

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

is in reduced echelon form.

x_1, x_2, x_3 are all free variables.

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad s, t, u \in \mathbb{R}.$$

So, the solution set is all of $\mathbb{R}^3$.

Prob 27 (pg 48)

$A\bar{x}=\bar{b}$ is consistent. Explain why it has a unique solution precisely when $A\bar{x}=\bar{0}$ has only the trivial solution.

$A\bar{x}=\bar{0}$ has only the trivial solution if A has no free variables.

We could use the same set of EROs that take A to echelon form, and apply them to $[A|\bar{b}]$. Hence, the system $A\bar{x}=\bar{b}$ has free variables if and only if A has free variables.

A similar question

Let $A\bar{x}=\bar{0}$ have nontrivial solutions. Can you guarantee that $A\bar{x}=\bar{b}$ always has infinitely many solutions?

The statement holds as long as $A\bar{x}=\bar{b}$ is consistent.

$$A=\begin{array}{cc} & x_2 \\ \end{array}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

x_2 free. So $A\bar{x}=\bar{0}$ has nontrivial solutions.

But $A\bar{x}=\begin{bmatrix}1\\1\end{bmatrix}$ is inconsistent!

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

When answering True/False problems, try to provide the simplest counter examples when possible, as above.