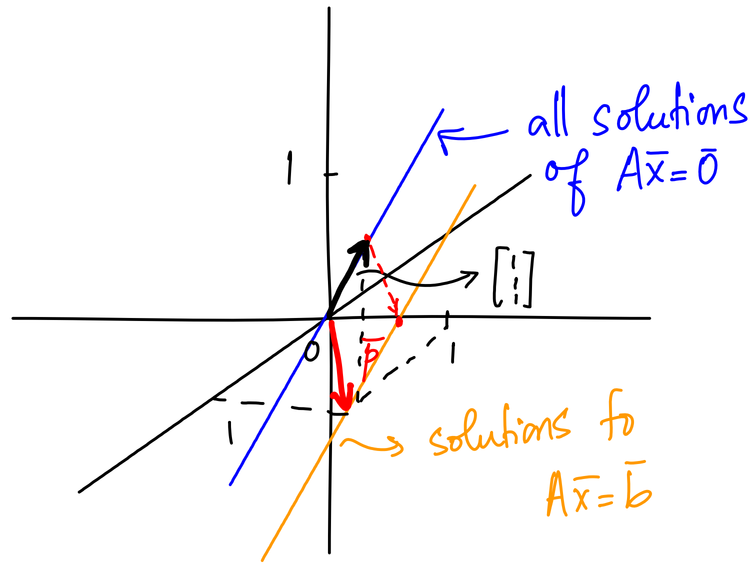
MATH 220- Lecture 7 (09/10/2013)

Solutions of $A\bar{x}=\bar{b}$ for nonzero $\bar{b}$ in terms of solutions to $A\bar{x}=\bar{0}$.



$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 3 \\ 2x_1 + x_2 - 3x_3 &= 3 \\ -x_1 + x_2 &= 0 \end{aligned}$$

Here $\bar{b}=\begin{bmatrix}3\\3\\0\end{bmatrix}$.

Previously, we had $\bar{b}=\bar{0}$.

In Lecture 6, we solved the corresponding homogeneous system, and visualized its solutions in parametric vector form.

We now repeat the same EROs on just $\bar{b}=\begin{bmatrix}3\\3\\0\end{bmatrix}$.

$$\begin{bmatrix}3\\3\\0\end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix}3\\-3\\3\end{bmatrix} \xrightarrow{R_3+R_2} \begin{bmatrix}3\\-3\\0\end{bmatrix} \xrightarrow{R_2\times(-\frac{1}{3})} \begin{bmatrix}3\\1\\0\end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix}1\\1\\0\end{bmatrix}$$

The reduced echelon form of $[A|\bar{b}]$ is hence

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \underbrace{x_1 \quad x_2}_{\text{basic}} \quad \underset{\text{free}}{x_3} \qquad \end{array}$$

$$\left.\begin{aligned} x_1 - x_3 &= 1 \\ x_2 - x_3 &= 1 \end{aligned}\right., \ x_3 \text{ free} \Bigg\} \text{ i.e., } \underbrace{\begin{aligned} x_1 &= 1+s \\ x_2 &= 1+s \end{aligned}, \ s \in \mathbb{R}}_{\text{parametric form}}$$

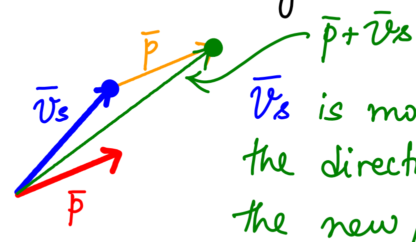
$$\bar{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \underset{\bar{p}}{\begin{bmatrix}1\\1\\0\end{bmatrix}} + \underset{\bar{v}}{\begin{bmatrix}1\\1\\1\end{bmatrix}} s, \quad s \in \mathbb{R} \Bigg\} \text{ parametric vector form}$$

$\bar{x} = \bar{v}s,\ s \in \mathbb{R}$ for $\bar{v} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ is the parametric vector form of solutions to $A\bar{x}=\bar{0}$.

$\boxed{\bar{x} = \bar{p} + \bar{v}s,\ s \in \mathbb{R}}$ is the parametric vector form for solutions to $A\bar{x}=\bar{b}$.

equation for a line through $\bar{p}$ parallel to $\bar{v}s$

Adding $\bar{p}$ to $\bar{v}s$ is equivalent to moving the vector $\bar{v}s$ in a direction along the line through origin and $\bar{p}$.



$\bar{v}s$ is moved along the direction of $\bar{p}$ to the new point $\bar{p}+\bar{v}s$

In fact, the above observation holds in the case of linear systems of equations in general, as long as the system in question is consistent.

Theorem If $A\bar{x}=\bar{b}$ has a solution $\bar{x}=\bar{p}$, then all solutions of $A\bar{x}=\bar{b}$ are given by $\bar{x} = \bar{p} + \bar{v}_h$, where $\bar{v}_h$ is any solution of $A\bar{x}=\bar{0}$.

'h' for homogeneous

Notice that the trivial solution corresponds to the choice $s=0$ for the homogeneous system. For the same value of the parameter in the case of the nonhomogeneous system, we get $\bar{x}=\bar{p}$ as the solution. So, the origin gets translated to $\bar{p}$.

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Describe and compare the solution sets of

$$x_1 - 2x_2 + 3x_3 = 0 \quad \text{and} \quad x_1 - 2x_2 + 3x_3 = 4.$$

$$\begin{array}{c} \\ \\ A = \end{array} \begin{array}{ccc} \text{basic} & \multicolumn{2}{c}{\overbrace{\qquad\quad}^{\text{free}}} \\ x_1 & x_2 & x_3 \\ \multicolumn{3}{c}{\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}} \end{array}$$

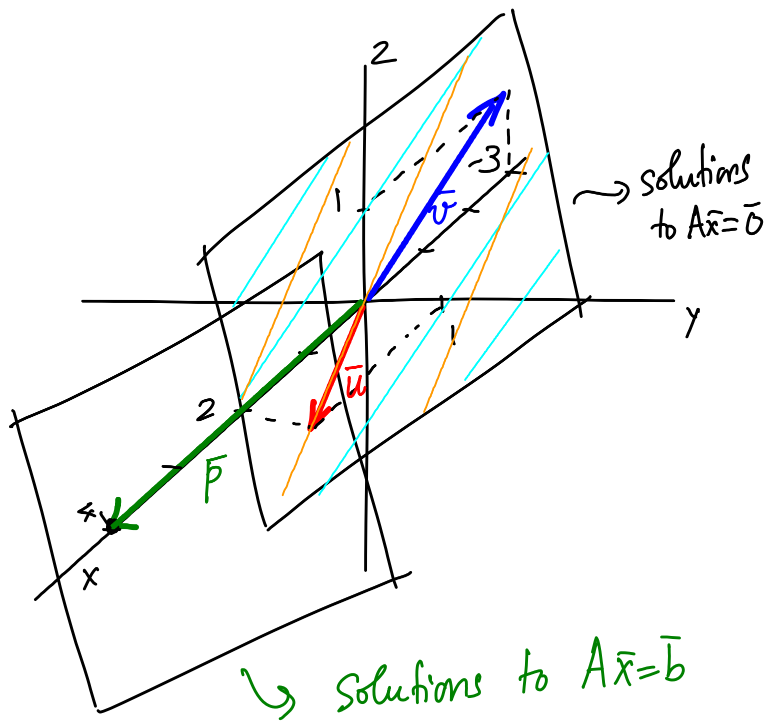
$x_1 = 2x_2 - 3x_3$, x_2, x_3 free

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{u}} s + \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}} t, \quad s, t \in \mathbb{R}$$

$$[A \mid \bar{b}] = \begin{bmatrix} 1 & -2 & 3 & | & 4 \end{bmatrix}$$

$$x_1 = 4 + 2x_2 - 3x_3$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}}_{\bar{p}} + \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{u}} s + \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\bar{v}} t$$



Solutions to $A\bar{x} = \bar{0}$ form a plane through $\bar{0}, \bar{u}, \bar{v}$. And the solutions to $A\bar{x} = \bar{b}$ form a parallel plane passing through $\bar{p}$.

Linear Independence (Section 1.7)

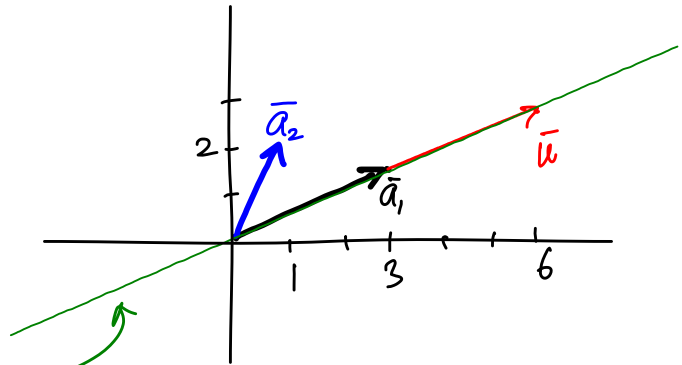
Recall

If $\bar{a}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\bar{a}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then

$\text{span}\{\bar{a}_1, \bar{a}_2\} = \mathbb{R}^2$.

But with $\bar{u} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$,

$\text{span}\{\bar{a}_1, \bar{u}\}$ is just the line through $\bar{0}$ and $\bar{a}_1$.



$\bar{a}_1$ and $\bar{a}_2$ are linearly independent here, i.e., they are not along the same line. While $\bar{a}_1$ and $\bar{u}$ are linearly dependent.

We now extend this idea of being "along the same line" (or not) to arbitrary collections of vectors in high dimensions.

the book uses p instead of n here.

Def The set $\{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ with each $\bar{v}_j \in \mathbb{R}^m$ is **linearly independent** (LI) if the vector equation
$\bar{v}_1 x_1 + \bar{v}_2 x_2 + \cdots + \bar{v}_n x_n = \bar{0}$ has only the trivial solution.

If there is a non-trivial solution, the set of vectors is **linearly dependent** (LD).

Since we already know how to check if $A\bar{x}=\bar{0}$ has only the trivial solution (when there are no free variables), we can use those results to directly answer questions about whether a given set of vectors is LI or not.

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$$A = \begin{bmatrix} 0 & -3 & 9 \\ 2 & 1 & -7 \\ -1 & 4 & -5 \\ 1 & -4 & -2 \end{bmatrix}.$$

Do the columns of A form a linearly independent set of vectors?

Equivalently, does $A\bar{x}=\bar{0}$ have only the trivial solution?

$$\begin{bmatrix} 0 & -3 & 9 \\ 2 & 1 & -7 \\ -1 & 4 & -5 \\ 1 & -4 & -2 \end{bmatrix} \xrightarrow{R_1 \rightleftarrows R_4} \begin{bmatrix} 1 & -4 & -2 \\ 2 & 1 & -7 \\ -1 & 4 & -5 \\ 0 & -3 & 9 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 9 & -3 \\ 0 & 0 & -7 \\ 0 & -3 & 9 \end{bmatrix} \xrightarrow{R_2 \rightleftarrows R_4} \begin{bmatrix} 1 & -4 & -2 \\ 0 & -3 & 9 \\ 0 & 0 & -7 \\ 0 & 9 & -3 \end{bmatrix}$$

$$\xrightarrow{R_4+3R_2} \begin{bmatrix} 1 & -4 & -2 \\ 0 & -3 & 9 \\ 0 & 0 & -7 \\ 0 & 0 & 24 \end{bmatrix} \xrightarrow{R_4+\frac{24}{7}R_3} \begin{bmatrix} 1 & -4 & -2 \\ 0 & -3 & 9 \\ 0 & 0 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

There are no free variables, and hence the system has only the trivial solution. So columns of A are LI.

We now describe several special cases of sets of vectors, for which we can determine linear (in)dependence more directly than by performing EROs.

Special Cases

1. $\{\bar{v}\}$ (Single vector).

   The set $\{\bar{v}\}$ is LI if $\bar{v} \neq \bar{0}$.

   To follow the definition, we are trying to find when does the system $\bar{v}x = \bar{0}$ have only the trivial solution. Naturally, when $\bar{v} \neq \bar{0}$, we can get the zero vector only by taking $x = 0$.

We will discuss three more special cases in the next lecture...