MATH 220 – Lecture 6 (09/05/2013)

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Theorem

The following statements are equivalent for $A \in \mathbb{R}^{m\times n}$, $\bar{x} \in \mathbb{R}^n$, $\bar{b} \in \mathbb{R}^m$.

1. $A\bar{x} = \bar{b}$ has a solution for each $\bar{b} \in \mathbb{R}^m$.
2. $\bar{b}$ is in the span of the columns of A.
3. Columns of A span $\mathbb{R}^m$.
4. **Every row of A has a pivot.**

We use this condition to check readily whether $A\bar{x} = \bar{b}$ is consistent

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Prob 22 pg 41

$\bar{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}$, $\bar{V}_3 = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix}$. Does $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ span $\mathbb{R}^3$? Why?

$$\text{Let } A = [\bar{v}_1\ \bar{v}_2\ \bar{v}_3] = \begin{bmatrix} 0 & 0 & 4 \\ 0 & -3 & -2 \\ -3 & 9 & -6 \end{bmatrix} \xrightarrow{R_3 \rightleftarrows R_1} \begin{bmatrix} -3 & 9 & -6 \\ 0 & -3 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

Every row of A has a pivot. So $\text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \mathbb{R}^3$.

In more detail, since each row of A has a pivot, $A\bar{x} = \bar{b}$ is consistent for every $\bar{b} \in \mathbb{R}^3$. Hence, every vector in $\mathbb{R}^3$ is in $\text{span}\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$.

Prob 30, pg 41

Construct a 3x3 matrix whose columns do not span $\mathbb{R}^3$. Justify.

for such problems, its best to create the "minimal", or simplist, example that works.

e.g., $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ works.

Similarly,

$B = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ also works.

At least one row should not have a pivot.

Prob 26 pg 41

$\bar{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$, $\bar{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$, and they satisfy $\underset{x_1}{2}\bar{u} \underset{x_2}{-3}\bar{v} - \bar{w} = \bar{0}$. $\bar{w}$

Find x_1, x_2 that satisfy $\begin{array}{c} \begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \\ \bar{u} \quad \bar{v} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \\ \bar{w} \end{array}$, without using EROs.

The given equation can be rewritten as $2\bar{u} + (-3)\bar{v} = \bar{w}$.
Hence $x_1 = 2$, $x_2 = -3$ satisfies $\bar{u}x_1 + \bar{v}x_2 = \bar{w}$, which is the given system.

How about finding a solution for $\begin{array}{c} \begin{bmatrix} 5 & 7 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} \\ \bar{w} \quad \bar{u} \end{array} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \\ \bar{v} \end{array}$, given the same relationship $2\bar{u} - 3\bar{v} - \bar{w} = \bar{0}$?

The given system is $\bar{w}x_1 + \bar{u}x_2 = \bar{v}$. To read off one solution, we rewrite $2\bar{u} - 3\bar{v} - \bar{w} = \bar{0}$ in this form, as follows.

$$\left(-\bar{w} + 2\bar{u} = 3\bar{v}\right) \times \tfrac{1}{3}$$

$$\Rightarrow \quad \underbrace{-\tfrac{1}{3}}_{x_1}\bar{w} + \underbrace{\tfrac{2}{3}}_{x_2}\bar{u} = \bar{v}.$$ Hence $x_1 = -\frac{1}{3}$, $x_2 = \frac{2}{3}$ is one solution.

We now study how to characterize when $A\bar{x} = \bar{b}$ is consistent for **all** $\bar{b}$. Naturally, we give this characterization in terms of A. To start with, we study the case when $\bar{b}$ is simplest, i.e., $\bar{b} = \bar{0}$.
zero vector

Homogeneous Systems of Linear Equations (Section 1.5)

$A\bar{x} = \bar{0}$ (all right-hand side entries are zero) is a **homogeneous system** of linear equations.

$\bar{x} = \bar{0}$ (the zero vector) is always a solution, and is called the **trivial solution**.

Q: Are there non-trivial solutions to $A\bar{x} = \bar{0}$?

Recall that a consistent system has either a unique solution, or has infinitely many solutions. For $A\bar{x} = \bar{0}$, the trivial solution is always present. Hence, if it has nontrivial solutions if it has infinitely many solutions, for which, it must have free variables.

A: There **are nontrivial solutions** if there is at least one **free variable**.

Prob 6, Pg 47

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ -x_1 + x_2 \quad &= 0 \end{aligned}$$

Does this system have nontrivial solutions? If yes, describe all of them.

$$\underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix}}_{A} \xrightarrow[R_3+R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \xrightarrow{R_3+R_2} \begin{array}{c} \overbrace{x_1 \quad x_2}^{\text{basic}} \quad \overset{\text{free}}{x_3} \\ \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

To have nontrivial solutions, there must exist at least one free variable. x_3 is free here, so the system does have non-trivial solutions.

We now describe all its solutions.

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \times \left(-\frac{1}{3}\right)} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} x_1 - x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$$

Hence, $\left\{ \begin{aligned} x_1 &= x_3 \\ x_2 &= x_3, \end{aligned} \quad x_3 \text{ free} \right\}$ describes all solutions

Equivalently, we can write $\left\{ \begin{aligned} x_1 &= s \\ x_2 &= s, \end{aligned} \quad \underset{\text{parameter}}{s} \in \mathbb{R} \right\}$, which is the parametric form.

We now represent the parametric form in an equivalent form involving a vector corresponding to the parameter s.

All solutions can be written in the vector form

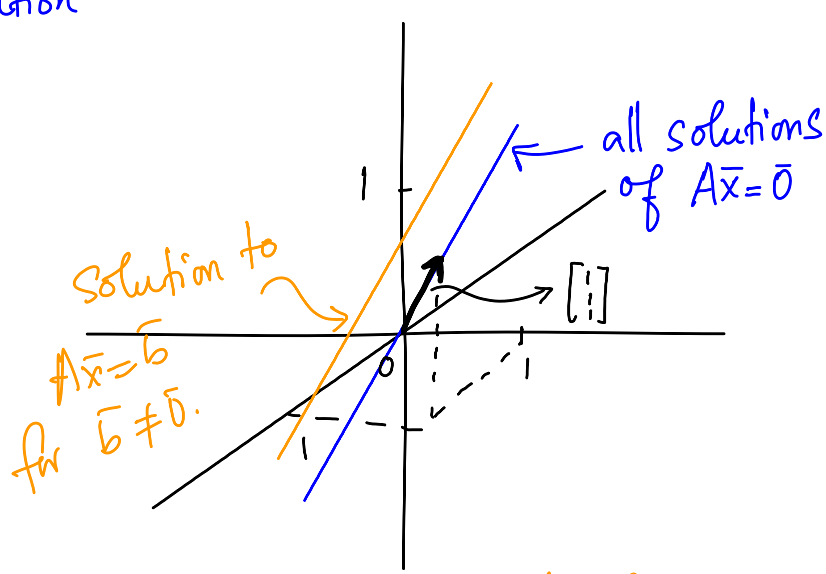
$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} x_3 \,, \quad \text{where } x_3 \text{ is free. Equivalently,}$$

$$\underbrace{\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} s}_{\text{parametric vector form of all solutions.}} \,, \quad s \in \mathbb{R}.$$

parameter that we can choose freely.

We can visualize the solutions as follows. All solutions form a line through the origin along the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ associated with the parameter s.

Notice that the trivial solution corresponds to $s=0$.



It turns out that the solutions for $A\bar{x}=\bar{b}$ for a nonzero $\bar{b}$ could be described as a parallel line, obtained by just "shifting" the solutions line for $A\bar{x}=\bar{0}$ by a vector. More on this picture in the next class...