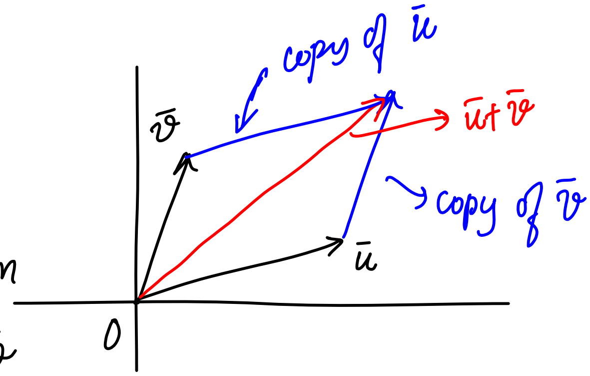
MATH 220 – Lecture 5 (09/03/2013)

Vector form of a system of linear equation:

$\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots + \bar{a}_n x_n = \bar{b}$

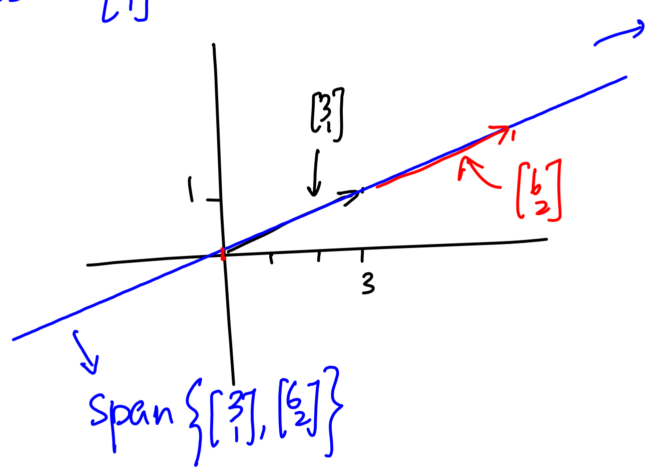
The system has a solution if $\bar{b}$ is a linear combination of the vectors $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$. Equivalently, if $\bar{b}$ is in span $\{\bar{a}_1, \dots, \bar{a}_n\}$. → the set of all linear combinations of $\bar{a}_1, \dots, \bar{a}_n$

The parallelogram rule of vector addition in $\mathbb{R}^2$ — the sum $\bar{u}+\bar{v}$ is the diagonal of the parallelgram formed by $\bar{u}, \bar{v}$. Equivalently, $\bar{u}+\bar{v}$ is the fourth vertex of the parallelogram formed by $\bar{0}$ (origin), $\bar{u}$, and $\bar{v}$.



Span $\left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}6\\2\end{bmatrix}\right\} = ?$ → notice that $\begin{bmatrix}6\\2\end{bmatrix}$ is $2 \times \begin{bmatrix}3\\1\end{bmatrix}$

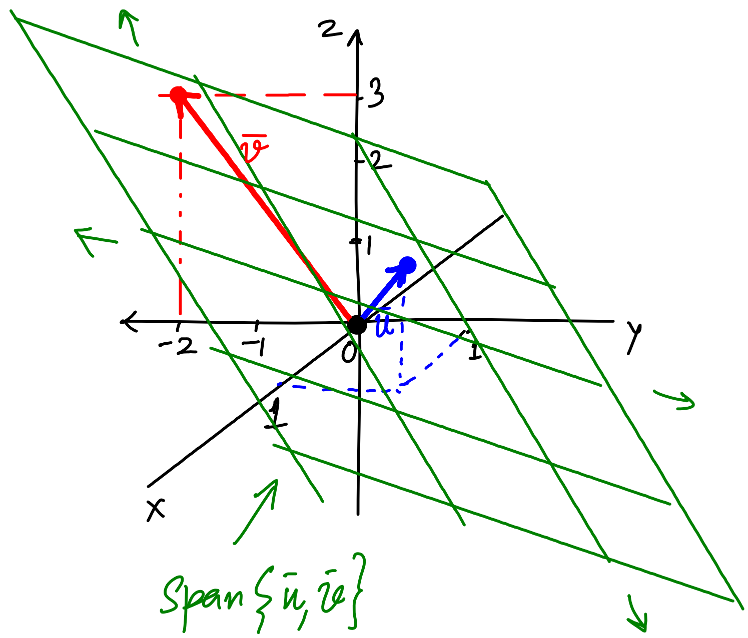
The span of $\begin{bmatrix}3\\1\end{bmatrix}$ and $\begin{bmatrix}6\\2\end{bmatrix}$ is the line through the origin and $\begin{bmatrix}3\\1\end{bmatrix}$.



$\text{Span}\{\bar{u}, \bar{v}\} = ?$ where

$$\bar{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} ?$$

$\{x_1 \bar{u} + x_2 \bar{v}\} = ?$
for all
$x_1, x_2 \in \mathbb{R}$



Span$\{\bar{u}, \bar{v}\}$ is the plane passing through $\bar{0}, \bar{u}, \bar{v}$.

Notice that in 3D space, 3 points that are not on a single straight line determine a plane uniquely. Imagine a sheet of paper passing through the three points, but extending without limits on all of its four edges.

This illustration of span$\{\bar{u}, \bar{v}\}$ also demonstrates the choice of the word "span". As such, span$\{\bar{u}, \bar{v}\}$ is also referred to as the plane generated by $\bar{u}$ and $\bar{v}$ (the origin $\bar{0}$ is understood to be included implicitly).

Prob 21, pg 32

$\bar{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in $\mathrm{Span}\{\bar{u}, \bar{v}\}$ for every h, k.

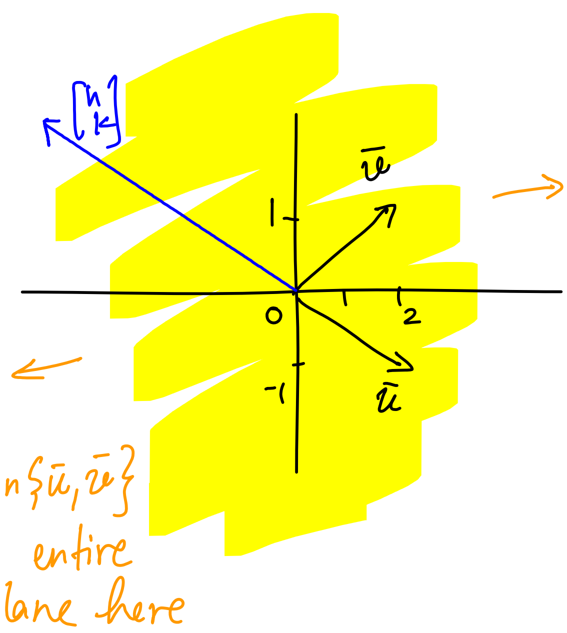
Show that $\bar{u}x_1 + \bar{v}x_2 = \begin{bmatrix} h \\ k \end{bmatrix}$ is consistent for all h, k.

$$\left[\begin{array}{cc|c} 2 & 2 & h \\ -1 & 1 & k \end{array}\right] \xrightarrow{R_1 \times \frac{1}{2}} \left[\begin{array}{cc|c} 1 & 1 & h/2 \\ -1 & 1 & k \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc|c} 1 & 1 & h/2 \\ 0 & 2 & k \end{array}\right]$$

The system is consistent for every h and k.

Here $\mathrm{Span}\{\bar{u}, \bar{v}\} = \mathbb{R}^2$

The span is all of $\mathbb{R}^2$.



Compare the span here to the span of $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \end{bmatrix}\right\}$ seen earlier, which was just a line through the origin. As in the case here, if the span of a set of vectors $\bar{a}_1, \dots, \bar{a}_n$ is all of the space in which the vectors sit, then life becomes easy. We know that the system

$$\bar{a}_1 x_1 + \dots + \bar{a}_n x_n = \bar{b}$$

is consistent for **every** $\bar{b}$!

Prob 16, pg 22

$\bar{u}=\begin{bmatrix}1\\0\\-2\end{bmatrix}$, $\bar{v}=\begin{bmatrix}-2\\1\\7\end{bmatrix}$, $\bar{y}=\begin{bmatrix}h\\-3\\-5\end{bmatrix}$. For what values of h is $\bar{y}$ in the plane generated by $\bar{u}$ and $\bar{v}$?

Equivalently, for what values of h is the system

$\bar{u}x_1+\bar{v}x_2=\bar{y}$ consistent?

$$\left[\begin{array}{cc|c}1&-2&h\\0&1&-3\\-2&7&-5\end{array}\right]\xrightarrow{R_3+2R_1}\left[\begin{array}{cc|c}1&-2&h\\0&1&-3\\0&3&-5+2h\end{array}\right]\xrightarrow{R_3-3R_2}\left[\begin{array}{cc|c}1&-2&h\\0&1&-3\\0&0&4+2h\end{array}\right]$$

$4+2h=0$ for a consistent system.

So, $h=-2$.

The matrix form $A\bar{x}=\bar{b}$ (Section 1.4)

We have already seen this form! For instance,

$$3x_1+x_2=7$$
$$x_1+2x_2=4$$

augmented matrix $\left[\begin{array}{cc|c}3&1&7\\1&2&4\end{array}\right]$

2×3 matrix

$\begin{bmatrix}3\\1\end{bmatrix}x_1+\begin{bmatrix}1\\2\end{bmatrix}x_2=\begin{bmatrix}7\\4\end{bmatrix}$ } vector equation

We now write it in matrix form:

$$\underbrace{\begin{bmatrix}3&1\\1&2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{\bar{x}}=\underbrace{\begin{bmatrix}7\\4\end{bmatrix}}_{\bar{b}}$$

where $A=\begin{bmatrix}3&1\\1&2\end{bmatrix}$ (2×2), $\bar{x}=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ (variable vector), $\bar{b}=\begin{bmatrix}7\\4\end{bmatrix}$ (right-hand side (rhs) vector).

$A\bar{x}$ is a **matrix-vector product.**

Let $A = [\bar{a}_1 \bar{a}_2 \cdots \bar{a}_n]$ be an $m \times n$ matrix. So, $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ are all m-vectors.

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ is an } n\text{-vector.}$$

Then $A\bar{x} = \bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots + \bar{a}_n x_n$ is the linear combination of the columns of A with the entries in $\bar{x}$ as scalars or weights.

$A\bar{x} = \bar{b}$ has a solution if and only if

the vector equation $\bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots + \bar{a}_n x_n = \bar{b}$ has a solution, which happens if and only if the system represented by the augmented matrix $[A \mid \bar{b}]$ has a solution.

We now discuss a condition that guarantees $A\bar{x} = \bar{b}$ has a solution, given in terms of the existence of pivots in each row of A. This condition is independent of $\bar{b}$.

Prob 16 pg 40

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 2 & 0 \\ 4 & -1 & 3 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

$\bar{a}_1 \quad \bar{a}_2 \quad \bar{a}_3$

Show that $A\bar{x}=\bar{b}$ is not consistent for all $\bar{b}$. Describe the collection of $\bar{b}$ for which it is consistent.

$$[A|\bar{b}] = \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2+2R_1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2+2b_1 \\ 0 & 7 & 7 & b_3-4b_1 \end{array}\right] \xrightarrow{R_3+\frac{7}{2}R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2+2b_1 \\ 0 & 0 & 0 & b_3-4b_1+\frac{7}{2}(b_2+2b_1) \end{array}\right]$$

$= 0$ for system to be consistent

i.e., $3b_1+\frac{7}{2}b_2+b_3=0$, i.e., $6b_1+7b_2+2b_3=0$.

Hence, the system is not consistent for all $\bar{b}$, but only for those $\bar{b}\in\mathbb{R}^3$ that satisfy $6b_1+7b_2+2b_3=0$.

The set of all $\bar{b}$ for which $A\bar{x}=\bar{b}$ is consistent is a plane through the origin described by $6b_1+7b_2+2b_3=0$.

Equivalently, $\text{span}\{\bar{a}_1,\bar{a}_2,\bar{a}_3\}$ is not all of $\mathbb{R}^3$, but a plane through origin sitting in $\mathbb{R}^3$.