MATH 220 – Lecture 4 (08/29/2013)

Echelon form and reduced echelon form

We now introduce notation using which we describe matrices in the forms in general – without writing the actual numbers.

Standard notation: $\begin{cases} \blacksquare \rightarrow \text{nonzero number} \\ * \rightarrow \text{zero or nonzero} \end{cases}$

e.g., $\begin{bmatrix} \blacksquare & * & * & 0 \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & * \end{bmatrix}$ is in echelon form

The above matrix is a 3×4 matrix (read as 3 "by" 4), which denotes its **size**. The size of matrix tells us how big it is.

In general, the **size** of a matrix is given as **(# rows) x (# columns).**

"by"

Similarly,

$\begin{bmatrix} 1 & \blacksquare & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is in reduced echelon form.

Notice that the entry denoted by $\blacksquare$ is not a pivot. As such, it could be zero or nonzero – in either case, the matrix is in reduced echelon form.

$\begin{bmatrix} \blacksquare & 0 & * & 0 \\ 0 & 0 & \blacksquare & * \\ * & 0 & 0 & 1 \end{bmatrix}$ is not in echelon form, though. When we are talking about such general forms, we consider all possible values for $*$ – so, when $*$ in the bottom left is indeed $\neq 0$, the matrix is not in echelon form.

Solution of linear systems

We can use row reduction to solve linear systems.

* form the augmented matrix.
* reduce to echelon form.
  - if the echelon form has a row of the form $[0 \cdots 0 | * \neq 0]$, the system is inconsistent.
  - if it does not have such a row, convert the matrix to reduced echelon form, and describe the solution(s). We illustrate this step on examples now.

Prob 10, pg 22

$$\begin{bmatrix} 1 & -2 & -1 & 4 \\ -2 & 4 & -5 & 6 \end{bmatrix} \xrightarrow{R_2+2R_1} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ 0 & 0 & -7 & 14 \end{array}\right] \xrightarrow{R_2 \times -1/7} \left[\begin{array}{ccc|c} 1 & -2 & -1 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow{R_1+R_2}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

pivot columns

x_1, x_3 are basic variables

x_2 is free

Def The variables corresponding to pivot columns in the augmented matrix are called **basic variables**. The remaining variables are called **free variables**. also called nonbasic variables

Idea: Describe (all) solution(s) by expressing the basic variables in terms of the free variables.

$$\left.\begin{aligned} x_1 - 2x_2 &= 2 \\ x_3 &= -2 \end{aligned}\right\} \text{ so, } \quad \boxed{\begin{aligned} x_1 &= 2 + 2x_2, \quad x_2 \text{ free} \\ x_3 &= -2 \end{aligned}}$$

Here, the value of x_3 does not depend on x_2.

Here is another example.

Prob 13, pg 22

Augmented matrix is given. Solve the corresponding system.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The augmented matrix is in echelon form, and does not have a row of the form $[0\ 0 \ldots 0 \mid * \neq 0]$. Hence the system is consistent.

Notice that x_1, x_2, x_4 are basic, and x_3, x_5 are free.

$$\left[\begin{array}{ccccc|c} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+3R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1 & -12 & 1 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+R_3} \underbrace{\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{reduced echelon form}}$$

$$\left.\begin{aligned} x_1 - 3x_5 &= 5 \\ x_2 - 4x_5 &= 1 \\ x_4 + 9x_5 &= 4 \end{aligned}\right\} \quad \underbrace{\begin{aligned} x_1 &= 5 + 3x_5 \\ x_2 &= 1 + 4x_5, \quad x_3, x_5 \text{ free} \\ x_4 &= 4 - 9x_5 \end{aligned}}_{\text{also called the parametric solution}}$$

x_3 and x_5 are parameters that can be chosen freely.

One can notice that all coefficients of x_3 are zero. Hence, we could just leave out x_3 from the discussion, without affecting the rest of the solution. At the same time, one should **not** assume that $x_3 = 0$, which is effectively what you are doing if you leave it out! Note that x_3 can assume **any** value, and hence is included as a parameter along with x_5, as one would do by default.

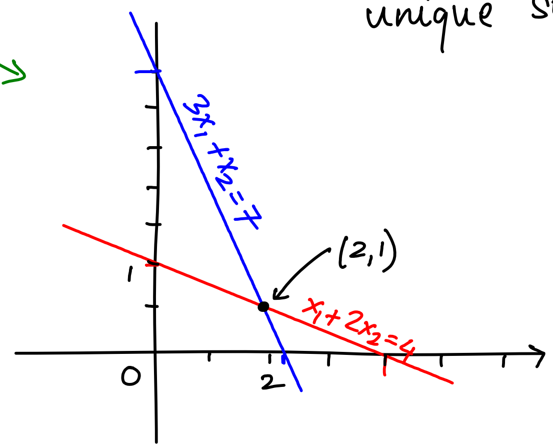
Vector Equations (Section 1.3)

$3x_1 + x_2 = 7$
$x_1 + 2x_2 = 4$

$$\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \end{bmatrix} \xrightarrow{\text{EROs}} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$x_1 = 2,\ x_2 = 1$ is the unique solution.

This is the "row picture." We plotted each row as a line, and the unique solution is the point of intersection of these lines.



We now talk about the "column picture."

$$\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \end{bmatrix} \text{ corresponds to } \begin{bmatrix} 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.$$

The columns here are called **vectors**. In fact, they are 2-vectors. (we specify the # entries as the size).

The set of all 2-vectors is denoted by $\mathbb{R}^2$ ("R-two")

real entries

two of them

$\mathbb{R}^n$: set of all n vectors with real entries.

e.g., $\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ is an n-vector.

the "bar" specifies that it's a vector!

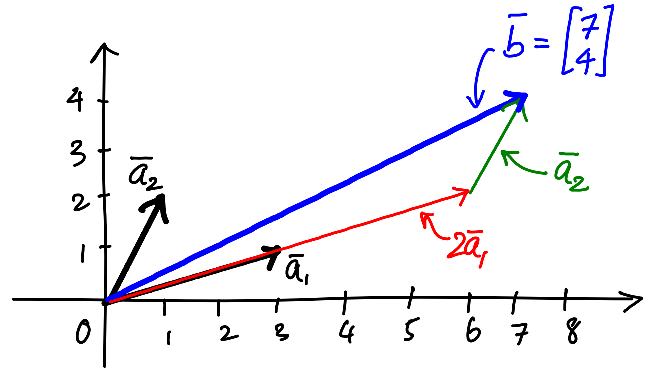
u_j (without the "bar") is a scalar for each $j = 1, 2, \ldots, n$.

We can plot the vectors (in 2D, and in 3D).

$$\begin{bmatrix}3\\1\end{bmatrix}x_1 + \begin{bmatrix}1\\2\end{bmatrix}x_2 = \begin{bmatrix}7\\4\end{bmatrix} \text{ (with } x_1=2, x_2=1\text{)}.$$

We scale the vector $\begin{bmatrix}3\\1\end{bmatrix}$ by the number x_1, and similarly scale $\begin{bmatrix}1\\2\end{bmatrix}$ by x_2, and add these two resulting vectors, and we should get $\begin{bmatrix}7\\4\end{bmatrix}$.

$$\begin{bmatrix}3\\1\end{bmatrix}2 + \begin{bmatrix}1\\2\end{bmatrix}1 = \begin{bmatrix}7\\4\end{bmatrix}$$



Adding scalar multiples of vectors in this fashion is called taking a **linear combination**.

Def If $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n$ all are m-vectors, then $\bar{w} = c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_n\bar{v}_n$, where c_i's are scalars, is a **linear combination** of $\bar{v}_1, \dots, \bar{v}_n$.

The set of all linear combinations is denoted as the **Span** of the vectors.

Denoting $\bar{a}_1 = \begin{bmatrix}3\\1\end{bmatrix}$, $\bar{a}_2 = \begin{bmatrix}1\\2\end{bmatrix}$, $\text{Span}\{\bar{a}_1, \bar{a}_2\} = \{$all vectors of the form $c_1\bar{a}_1 + c_2\bar{a}_2$, for scalars $c_1, c_2\}$.

e.g., $-3\bar{a}_1 + 4\bar{a}_2 = -3\begin{bmatrix}3\\1\end{bmatrix} + 4\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-3\times3+4\times1\\-3\times1+4\times2\end{bmatrix} = \begin{bmatrix}-5\\5\end{bmatrix}$.

So, $\begin{bmatrix}-5\\5\end{bmatrix}$ is a vector in $\text{span}\{\bar{a}_1, \bar{a}_2\}$.

We have already seen questions about when a system of linear equations has solutions, or not. The same questions could be raised in the context of vectors, their span, and linear combinations. Here is an illustration.

Q: Is $\begin{bmatrix} 8 \\ 3 \end{bmatrix}$ in $\text{span}\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$?

Equivalently, are there scalars x_1, x_2 such that

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 8 \\ 3 \end{bmatrix} ?$$

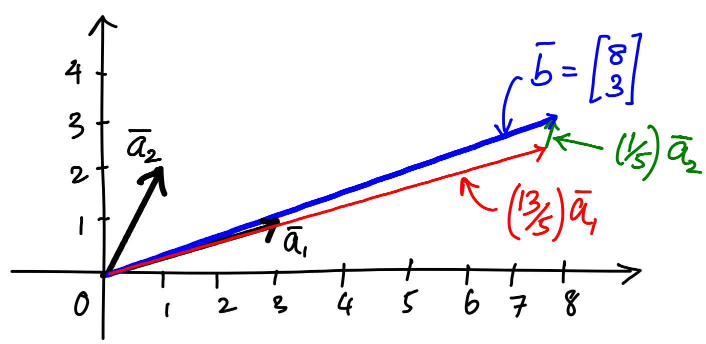
But this question is the same as the following one:

Does the system $\begin{bmatrix} 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$ have a solution?

Or, equivalently, does the system $\left\{\begin{matrix} 3x_1 + x_2 = 8 \\ x_1 + 2x_2 = 3 \end{matrix}\right\}$ have a solution?

$$\left[\begin{array}{cc|c} 3 & 1 & 8 \\ 1 & 2 & 3 \end{array}\right] \xrightarrow[\text{and then } R_1 \rightleftarrows R_2]{R_1 - 3R_2} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & -5 & -1 \end{array}\right] \xrightarrow{R_2 \times -1/5}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 1/5 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 13/5 \\ 0 & 1 & 1/5 \end{array}\right]$$



$x_1 = \frac{13}{5}$, $x_2 = \frac{1}{5}$ is the unique solution.

Hence, $\bar{b} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$ is in $\text{span}\{\bar{a}_1, \bar{a}_2\}$, where $\bar{a}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\bar{a}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.