MATH 220 – Lecture 30 (12/05/2013)

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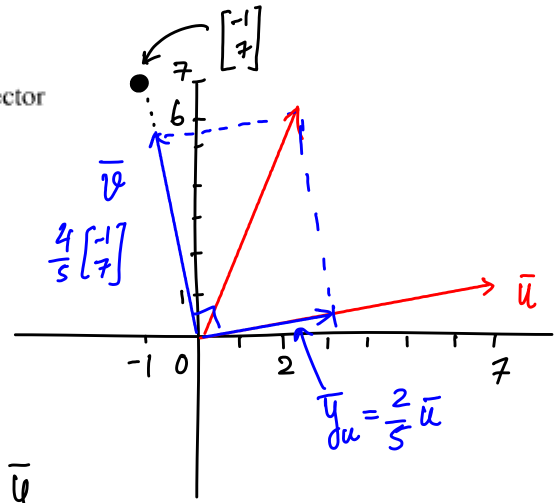
14. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span $\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.



Write $\bar{y} = \bar{y}_u + \bar{v}$ where

$\bar{y}_u = \alpha \bar{u}$ and $\bar{y}_u \perp \bar{v}$

Can find the orthogonal projection of $\bar{y}$ on to $\bar{u}$ to get $\bar{y}_u$.

$$\alpha = \frac{\bar{y}\cdot\bar{u}}{\bar{u}\cdot\bar{u}} = \frac{\begin{bmatrix} 2 \\ 6 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ 1 \end{bmatrix}}{(7)^2+(1)^2} = \frac{2\times 7 + 6\times 1}{49+1} = \frac{20}{50} = \frac{2}{5}$$

$\begin{bmatrix} 7 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$$\bar{y}_u = \frac{2}{5}\bar{u} = \frac{2}{5}\begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

$$\bar{v} = \bar{y} - \bar{y}_u = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{5}{5}\begin{bmatrix} 2 \\ 6 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 \\ 28 \end{bmatrix} = \frac{4}{5}\begin{bmatrix} -1 \\ 7 \end{bmatrix}.$$

So, $$\bar{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \frac{2}{5}\begin{bmatrix} 7 \\ 1 \end{bmatrix} + \frac{4}{5}\begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

are orthogonal!

as $\begin{bmatrix} 7 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 7 \end{bmatrix} = -7+7 = 0$

Properties of scalar products

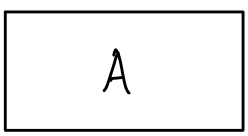
$$\bar{u}\cdot(\bar{v}+\bar{w}) = \bar{u}\cdot\bar{v} + \bar{u}\cdot\bar{w}$$

$$\bar{u}\cdot(c\,\bar{v}) = (c\,\bar{u})\cdot\bar{v} = c\,(\bar{u}\cdot\bar{v})$$

Review of final

5. Justify your answer for each of the following.

(a) If A has more columns than rows, can the columns of A be linearly independent?



No.
All columns cannot be pivot columns.
OR There will be free variables.

(b) If A is a 5×5 matrix and the rank of A is 5, is $\det(A) = 0$?

No. A has five 5 pivot columns and no free variables.
So A is invertible. So $\det(A) \neq 0$.

(c) Do six linearly independent vectors in $\mathbb{R}^9$ span a subspace of dimension six?

dimension = # vectors in any basis
The six LI vectors form a basis for the span of these vectors. Hence the answer is YES.

But they do **NOT** span $\mathbb{R}^6$, as each vector is in $\mathbb{R}^9$ to start with.

(d) If A, B, and C are $n \times n$ matrices and $AB = AC$, must $B = C$?

No. $B = C$ only if A is invertible.

$A^{-1}(AB = AC)$

$\underbrace{(A^{-1}A)}_{I} B = \underbrace{(A^{-1}A)}_{I} C$ or $B = C$

7. Suppose that A is matrix with $\text{rank}(A) = 3$, $\dim \text{Nul}(A) = 2$, and such that the row reduced echelon form of A has one row of zeros. How many rows does A have? How many columns does A have?

rank theorem says $\text{rank}(A) + \dim(\text{Nul}(A)) = n$ when A is $m \times n$

Here $n = 3 + 2 = 5$

Since $\text{rank}(A) = 3$, there are three pivots. So there should be 3 nonzero rows in $\text{rref}(A)$. Since there is one zero row in $\text{rref}(A)$, A has $3 + 1 = 4$ rows.

9. Let $x \in \mathbb{R}^n$ be an eigenvector of both the $n \times n$ matrices A and B. Show that x is an eigenvector of the matrix AB.

Let λ, μ be the eigenvalues of A and B corresponding to the eigenvector $\bar{x}$. So

$A\bar{x} = \lambda \bar{x}$ So $AB\bar{x} = A(B\bar{x}) = A(\mu\bar{x}) = \mu(A\bar{x})$

$B\bar{x} = \mu \bar{x}$ $\quad = \mu(\lambda\bar{x}) = \lambda\mu\bar{x}$

So $\bar{x}$ is an eigenvector of AB for the eigenvalue $\lambda\mu$.

Practice final

7. **(7)** Construct a *nonzero* 3×3 matrix A with rank 2, and a vector $\mathbf{b}$ that is *not* in Nul A.

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ has rank 2.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3+R_1]{R_2+R_1} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix} \text{ works.}$$

$\bar{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not in Nul A, as

$$A\bar{b} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq \bar{0}.$$

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ also works. Or, you could directly write down $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, for instance. There are two pivots in the matrix here, and all entries are nonzero.

The same $\bar{b}$ given above works in both cases here as well, since $A\bar{b} \neq \bar{0}$ in both cases.

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(10) Let $A + B$ and C be $n \times n$ invertible matrices. Solve the following equation for X. Justify each step in your solution.

$$C^{-1}(XB + XA)C = C^T.$$

$$C^{-1}(X(B+A))C = C^T \qquad \text{as } A(B+C) = AB + AC$$

$$\underbrace{C\,C^{-1}}(X(B+A))\underbrace{C\,C^{-1}} = C\,C^T C^{-1} \qquad C \text{ is invertible}$$

$$I\,(X(B+A))\,I = C\,C^T C^{-1} \qquad \text{as } CC^{-1} = I$$

$$X\underbrace{(B+A)(B+A)^{-1}}_{I} = C\,C^T C^{-1}(B+A)^{-1} \qquad A+B = B+A$$

$$A+B \text{ invertible.}$$

$$X = C\,C^T C^{-1}(A+B)^{-1}.$$

Notice that $C^T C^{-1} \neq I$! We had seen earlier that $(C^{-1})^T = (C^T)^{-1}$. But here we have $C^T C^{-1}$, which cannot be simplified further.