MATH 220 – Lecture 2 (08/22/2013)

Recall Dude's problem:

$$x_1 + x_2 = 5 \quad \text{—— (1)}$$
$$8x_1 + 16x_2 = 48 \quad \text{—— (2)}$$

$$\boxed{\begin{aligned} x_1 \quad &= 4 \\ x_2 &= 1 \end{aligned}}$$

this is the unique solution, but could also be viewed as a system of two linear equations.

In general, we can have any number of equations in any number of variables. To solve the system, we go to an "easier" system using operations that preserve the solutions.

Goal: Eliminate x_1 from equations (2), (3), ..
eliminate x_2 from equations (1), (3), ...
⋮

We illustrate this procedure below.

$(2) + -8\times(1)$

we replace equation (2) by the sum of itself and $(-8)\times$(equation 1).

$$x_1 + x_2 = 5 \quad \text{—— (1)}$$

$$\left[\begin{aligned} 8x_1 + 16x_2 &= 48 \quad + \\ -8x_1 - 8x_2 &= 5\times -8 \end{aligned}\right] \longrightarrow 8x_2 = 8$$

$$\begin{aligned} x_1 + x_2 &= 5 \quad \text{—— (1)} \\ 8x_2 &= 8 \quad \text{—— (2')} \end{aligned}$$

$(2')\times\frac{1}{8}$

$$\begin{aligned} x_1 + x_2 &= 5 \quad \text{—— (1)} \\ x_2 &= 1 \quad \text{—— (2'')} \end{aligned}$$

$(1) - (2'')$

$$\begin{aligned} x_1 \quad &= 5-1 = 4 \quad \text{—— (1')} \\ x_2 &= 1 \quad \text{—— (2'')} \end{aligned}$$

This procedure of transforming the original system to an equivalent system is called **Gaussian elimination**.

Def Two systems are **equivalent** if they have the same set of solutions.

↓
"Definition"

Matrix Notation

We present a much more compact representation of these operations — by working just with the numbers!

A **matrix** is a rectangular array of numbers. It has rows and columns.

e.g., $A = \begin{bmatrix} 1 & 1 \\ 8 & 16 \end{bmatrix}$ is the matrix of coefficients.

A **vector** is a single row or column of numbers.

(lower case letters with the bar are vectors, e.g., $\bar{a}, \bar{\beta}, \bar{x}, \bar{y}$, etc.)

"bar"

e.g., $\bar{b} = \begin{bmatrix} 5 \\ 48 \end{bmatrix}$ is the rhs (right-hand side) vector.

Augmented matrix for a system → attach the rhs vector to the matrix of coefficients. → represents the entire system.

this line is a "separator" — we use it just for convenience.

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 8 & 16 & 48 \end{array}\right] \text{ or, in general, } [A \mid \bar{b}]$$

We perform the permitted operations on the augmented matrix. These operations are called

elementary row operations (EROs)

do not change the solutions

work with the rows of $[A|\bar{b}]$, or on any matrix A in general.

It is important to remember that EROs can be applied to **any** matrix, and not just to augmented matrices. When applied to an augmented matrix, we are working with the equations in that system.

Each row in $[A|\bar{b}]$ is one equation.

There are three types of EROs.

1. **Replacement**: Replace a row with the sum of itself and a multiple of another row.

2. **Interchange**: swap two rows.

3. **Scaling**: multiply a row by a **nonzero** number.

If you multiply by zero, you're removing that equation! Don't do that!!

In the next (few) lecture(s), we will formalize the ideas for how to choose the EROs we would apply. For now, we will guess — the goal is to simplify, by eliminating x_1 from rows 2, 3, …, x_2 from rows 1, 3, …, and so on.

To zero out the 8 in Row 2, we could use a replacement ERO. Then we do a scaling ERO, and so on.

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 8 & 16 & 48 \end{array}\right] \xrightarrow{R_2 - 8R_1} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 8 & 48-8\times 5 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 8 & 8 \end{array}\right] \xrightarrow{R_2 \times \frac{1}{8}} \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

notation for replacement EROs

notation for Scaling ERO

For an interchange ERO, we use the following notation: $R_1 \rightleftarrows R_2$ (for swapping rows 1 & 2, for instance).

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Solve
$$\begin{aligned} x_1 - 5x_2 + 4x_3 &= -3 \\ 2x_1 - 7x_2 + 3x_3 &= -2 \\ -2x_1 + x_2 + 7x_3 &= -1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{array}\right] \xrightarrow{R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 0 & -6 & 10 & -3 \\ -2 & 1 & 7 & -1 \end{array}\right] \xrightarrow{R_3 + 2R_1} \left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 0 & -6 & 10 & -3 \\ 0 & -9 & 15 & -7 \end{array}\right]$$

$$\xrightarrow{R_2 \times \left(\frac{-1}{6}\right)} \left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 0 & 1 & -10/6 & 1/2 \\ 0 & -9 & 15 & -7 \end{array}\right] \quad \frac{-3}{-6} \quad \xrightarrow{R_3 + 9R_2} \left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 0 & 1 & -10/6 & 1/2 \\ 0 & 0 & 0 & -5/2 \end{array}\right]$$

For simple fractions as seen here, it's best to stick with them as is, rather than go to decimal notation.

$$0x_1 + 0x_2 + 0x_3 = -5/2$$

which cannot be true!

Hence the system is inconsistent, i.e., it has no solutions.

Whenever you get a row of the form [0 0 … 0 | *], where * is nonzero, the system is inconsistent.

So, as long as you do not see such a row, the system is consistent. It can have a unique solution, or infinitely many solutions.

Def Two matrices are **row equivalent** if there is a series of EROs that transforms one matrix into the other.

Note Every ERO is reversible, i.e., for every ERO, there is a complementary ERO that reverses its effect.

e.g., consider the first ERO from the previous problem. The complementary ERO is shown here.

$$\left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & -2 \\ -2 & 1 & 7 & -1 \end{array}\right] \underset{R_2 - R_3}{\overset{R_2 + R_3}{\rightleftarrows}} \left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 0 & -6 & 10 & -3 \\ -2 & 1 & 7 & -1 \end{array}\right]$$

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Given the matrix $\begin{bmatrix} 1 & h & -5 \\ 2 & -8 & 6 \end{bmatrix}$, find the values of h so that it is the augmented matrix of a consistent system.

$$\begin{bmatrix} 1 & h & -5 \\ 2 & -8 & 6 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & h & -5 \\ 0 & -8-2h & 16 \end{bmatrix}$$

cannot be [0 0 16], for the system to be consistent.

We need $-8-2h \neq 0$, i.e., $\boxed{h \neq -4}$.

You could write all values except -4, or simply put $h \neq -4$.