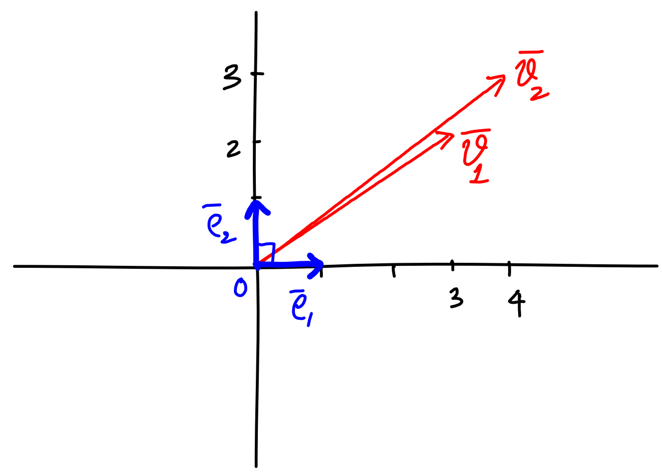
MATH 220 – Lecture 29 (12/03/2013)

Inner product, length (norm), orthogonality

Basis for a subspace:

Consider $\mathbb{R}^2$, for example. Let $\bar{v}_1 = \begin{bmatrix}3\\2\end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix}4\\3\end{bmatrix}$. Then

$\mathcal{B}_1 = \{\bar{v}_1, \bar{v}_2\}$ is a basis.



But $\mathcal{B}_e = \{\bar{e}_1, \bar{e}_2\} = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is also a basis for $\mathbb{R}^2$.

In fact, it is the standard basis.

Notice that $\bar{e}_1$ and $\bar{e}_2$ are perpendicular to each other, and are also of unit length. "most LI"

$\bar{v}_1$ and $\bar{v}_2$ are not collinear, and hence are LI, and so $\mathcal{B}_1 = \{\bar{v}_1, \bar{v}_2\}$ is indeed a basis for $\mathbb{R}^2$. But $\bar{e}_1$ and $\bar{e}_2$ are the "most LI", as they are the farthest away from being collinear.

We can extend these results to any subspace — given any basis for a subspace, we can find a basis that has vectors of unit length, and are "perpendicular" to each other.

Inner product (or scalar product or dot product)

$$\bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}_{n\times 1}, \quad \bar{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{n\times 1}. \quad \text{Then } \bar{u}^T\bar{v} = \underbrace{[u_1 \; u_2 \cdots u_n]}_{1\times n} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{n\times 1} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

$$\bar{u}\cdot\bar{v} = \bar{u}^T\bar{v} = \bar{v}^T\bar{u} = u_1v_1 + \cdots + u_nv_n.$$

"dot"

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Compute the quantities in Exercises 1–8 using the vectors

$$\mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

2. $\mathbf{w}\cdot\mathbf{w}$, $\mathbf{x}\cdot\mathbf{w}$, and $\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$

$$\bar{w}\cdot\bar{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 3\times 3 + (-1)\times(-1) + (-5)\times(-5)$$
$$= 3^2 + (-1)^2 + (-5)^2 = 35$$

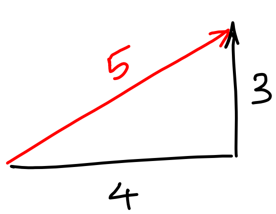
$$\bar{x}\cdot\bar{w} = 6\cdot 3 + (-2)(-1) + 3(-5)$$
$$= 18 + 2 - 15 = 5.$$

$$\frac{\bar{x}\cdot\bar{w}}{\bar{w}\cdot\bar{w}} = \frac{5}{35} = \frac{1}{7}.$$

Length of a vector (also called norm)

$$\|\bar{v}\| = \sqrt{\bar{v}\cdot\bar{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \qquad \left(\text{Notice that } \bar{v}\cdot\bar{v} = \|\bar{v}\|^2\right)$$

e.g., $\bar{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\qquad \|\bar{v}_2\| = \sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5.$



$(\text{hypotenuse})^2 = (\text{base})^2 + (\text{altitude})^2$

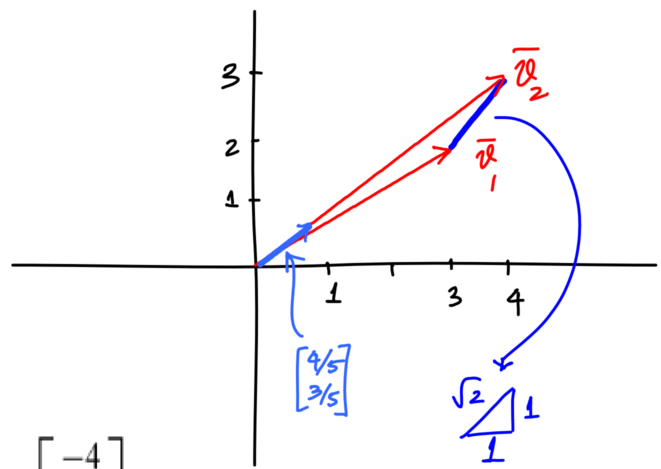
Thus, we are extending the Pythagorean theorem to higher dimensions here to define the length of a vector.

Distance between two vectors $\bar{u}, \bar{v}$

$$\text{dist}(\bar{u},\bar{v}) = \|\bar{u}-\bar{v}\| \quad \text{(length of the difference.)}$$

$$\text{dist}\left(\begin{bmatrix}3\\2\end{bmatrix},\begin{bmatrix}4\\3\end{bmatrix}\right) = \left\|\begin{bmatrix}3-4\\2-3\end{bmatrix}\right\|$$

$$= \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$$



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14. Find the distance between $\mathbf{u} = \begin{bmatrix}0\\-5\\2\end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix}-4\\-1\\8\end{bmatrix}$.

$$\text{dist}(\bar{u},\bar{z}) = \|\bar{u}-\bar{z}\| = \left\|\begin{bmatrix}0-{-4}\\-5-{-1}\\2-8\end{bmatrix}\right\| = \left\|\begin{bmatrix}4\\-4\\-6\end{bmatrix}\right\| = \sqrt{4^2+(-4)^2+(-6)^2}$$

$$= \sqrt{68}.$$

Unit vector in the direction of $\bar{v}$

$\hat{v} = \dfrac{\bar{v}}{\|\bar{v}\|}$ is a unit vector, i.e., vector of length 1, along $\bar{v}$.

$$\bar{v}_1 = \begin{bmatrix}3\\2\end{bmatrix} \quad \|\bar{v}_1\| = \sqrt{3^2+2^2} = \sqrt{13} \qquad \hat{v}_1 = \frac{1}{\sqrt{13}}\bar{v}_1 = \frac{1}{\sqrt{13}}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}3/\sqrt{13}\\2/\sqrt{13}\end{bmatrix}$$

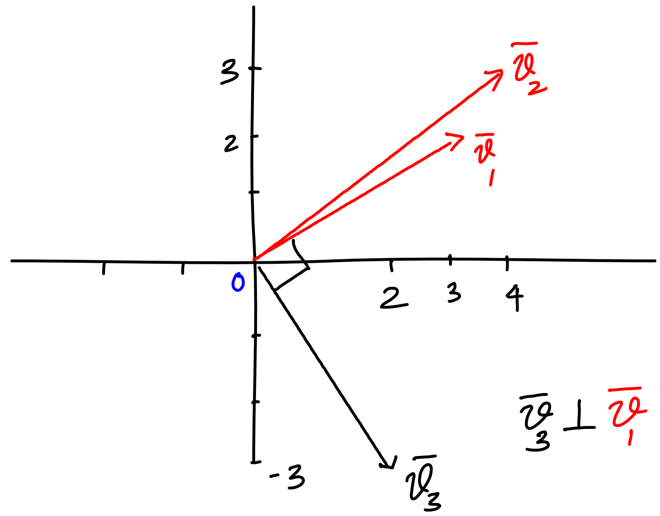
$$\bar{v}_2 = \begin{bmatrix}4\\3\end{bmatrix} \quad \|\bar{v}_2\| = 5 \qquad \hat{v}_2 = \frac{1}{5}\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}4/5\\3/5\end{bmatrix}$$

$\bar{u}$ and $\bar{v}$ (two vectors in $\mathbb{R}^n$) are **orthogonal** if $\bar{u}\cdot\bar{v} = 0$. We denote $\bar{u} \perp \bar{v}$.

e.g., $\bar{u} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\bar{v} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ $\quad \bar{u}\cdot\bar{v} = 3\cdot 0 + 0\cdot 4 = 0$

$\bar{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\bar{v}_3 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$\bar{v}_1 \cdot \bar{v}_3 = 3\times 2 + 2\times(-3) = 0.$



<u>Orthogonal sets</u>

$\{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_p\}$ in $\mathbb{R}^n$ is an orthogonal set if $\bar{u}_i \cdot \bar{u}_j = 0$ for all $i \neq j$, i.e., $\bar{u}_i \perp \bar{u}_j$ for $i \neq j$.

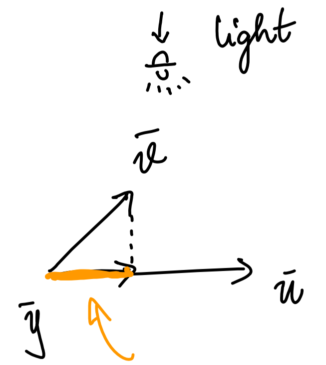
$\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set.

An orthogonal basis is a basis which is an orthogonal set. An **orthonormal** basis is an orthogonal basis where the vectors each have unit length.

$\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$.

And $\left\{ \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^3$.

Orthogonal projection



shadow of $\bar{v}$ on $\bar{u}$

The length of the "shadow of $\bar{v}$ along $\bar{u}$" is the length of $\bar{v}$ along $\bar{u}$.

The orthogonal projection of $\bar{v}$ on to $\bar{u}$ is the vector $\bar{y} = \alpha \bar{u}$ where $\alpha = \dfrac{\bar{v} \cdot \bar{u}}{\bar{u} \cdot \bar{u}}$.

Consider $\bar{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\bar{v}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ again. The orthogonal projection of $\bar{v}_2$ onto $\bar{v}_1$ is the vector

$$\bar{y} = \frac{\bar{v}_2 \cdot \bar{v}_1}{\bar{v}_1 \cdot \bar{v}_1} \bar{v}_1$$

We get

$$\bar{y} = \left(\frac{\begin{bmatrix} 4 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}}{\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}} \right) \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \left(\frac{4 \times 3 + 3 \times 2}{3^2 + 2^2} \right) \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \frac{18}{13} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 54/13 \\ 36/13 \end{bmatrix}$$