MATH 220 – Lecture 28 (11/21/2013)

Homework on Chapter 6 – due before the final
will count for 10 pts in the final

Final Exam: Tue, Dec 10, 7–9 pm
in Heald G3 (auditorium)
Will count for 90 pts in the final

The invertible matrix theorem (IMT)

a. $A \in \mathbb{R}^{n\times n}$ is invertible

⋮

s. zero is not an eigenvalue of A.

t. $\det A \neq 0$.

s. If λ is an eigenvalue, then $A\bar{x} = \lambda \bar{x}$ for some $\bar{x} \neq \bar{0}$. Hence if $\lambda = 0$ is an eigenvalue, $A\bar{x} = \bar{0}$ has a nontrivial solution. So A is not invertible.

The Characteristic Equation

$\det(A - \lambda I) = 0$ is the **characteristic equation** of A (in unknown λ).

The polynomial given by $\det(A-\lambda I)$ is called the **characteristic polynomial** of A.

28·2

Pg 279

Find the characteristic polynomial and the real eigenvalues of the matrices in Exercises 1–8.

4. $\begin{bmatrix} 8 & 2 \\ 3 & 3 \end{bmatrix} = A$

$$\det(A-\lambda I) = \begin{vmatrix} 8-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = (8-\lambda)(3-\lambda) - 2\times 3$$

$$= \lambda^2 - 11\lambda + 24 - 6$$

$$= \lambda^2 - 11\lambda + 18 \leftarrow \text{characteristic polynomial}$$

$\lambda^2 - 11\lambda + 18 = 0$ is the characteristic equation

$\Rightarrow (\lambda-2)(\lambda-9) = 0$ So, $\lambda = 2, 9$ are the eigenvalues.

(eigenvalues are solutions or roots of the characteristic equation)

Pg 279-280

Exercises 9–14 require techniques from Section 3.1. Find the characteristic polynomial of each matrix, using either a cofactor expansion or the special formula for 3×3 determinants described prior to Exercises 15–18 in Section 3.1. [*Note:* Finding the characteristic polynomial of a 3×3 matrix is not easy to do with just row operations, because the variable λ is involved.]

10. $\begin{bmatrix} 3 & 1 & 1 \\ 0 & 5 & 0 \\ -2 & 0 & 7 \end{bmatrix} = A$

Easiest to expand along a row/column with lots of zeros!

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ \boxed{0 \quad 5-\lambda \quad 0} \\ -2 & 0 & 7-\lambda \end{vmatrix} \longrightarrow \text{expand!}$$

$$= (-1)^{2+2}(5-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ -2 & 7-\lambda \end{vmatrix} = (5-\lambda)\left[(3-\lambda)(7-\lambda) - 1\times(-2)\right]$$

$$= (5-\lambda)\left[\lambda^2 - 10\lambda + 21 + 2\right] = 5\lambda^2 - 50\lambda + 115 - \lambda^3 + 10\lambda^2 - 23\lambda$$

$$= -\lambda^3 + 15\lambda^2 - 73\lambda + 115 \leftarrow \text{characteristic polynomial}$$

Def (Algebraic) **multiplicity** of an eigenvalue λ is the number of times it appears as a root of the characteristic equation.

pg 280

For the matrices in Exercises 15–17, list the real eigenvalues, repeated according to their multiplicities.

17. $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}$

The matrix is (lower) triangular, hence the eigenvalues are the entries in the diagonal.

The eigenvalues are 3, 3, 1, 1, 0.

Alternatively we can express each eigenvalue along with its multiplicity in braces, i.e., 3(2), 1(2), 0(1).

$\lambda=0$ appears as an eigenvalue once

Q: How many eigenvalues can $A \in \mathbb{R}^{n\times n}$ have?

Recall that eigenvalues are roots of the characteristic equation, which is $\det(A-\lambda I)=0$. In $A-\lambda I$, the n entries along the diagonal have λ in them. As such, $\det(A-\lambda I)$ is a polynomial of degree at most n. Hence A has at most n eigenvalues. But they need not all be distinct.

pg 280

18. It can be shown that the algebraic multiplicity of an eigenvalue λ is always greater than or equal to the dimension of the eigenspace corresponding to λ. Find h in the matrix A below such that the eigenspace for $\lambda = 4$ is two-dimensional:

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Reword: Find h such that $\dim(\mathrm{Nul}(A-\lambda I)) = 2$, i.e., $A-\lambda I$ has 2 free variables.

$$A-\lambda I = \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{bmatrix} \xrightarrow[R_4+\frac{1}{7}R_3]{R_2+R_1} \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 0 & h+3 & 6 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$h+3 = 0$ so that we have 2 pivots.

So $h=-3$ makes the eigenspace corresponding to the eigenvalue $\lambda = 4$ 2-dimensional.

Similar Matrices

Def Given $A, B \in \mathbb{R}^{n\times n}$, A is similar to B if there is an invertible matrix $P \in \mathbb{R}^{n\times n}$ such that $P^{-1}AP = B$.

In this case, $PBP^{-1} = A$, so B is similar to A.

So we just say that **A and B are similar**, and write $A \sim B$.

using P^{-1} as the invertible matrix, $(P^{-1})^{-1}B(P^{-1}) = A$.

We have already seen that if $A \sim B$, then

$\det A = \det B$, as

$A = P^{-1}BP$ where P is invertible.

So $\det A = \det(P^{-1}) \cdot \det B \cdot \det(P)$ as $\det AB = \det A \cdot \det B$

$= \dfrac{1}{\cancel{\det(P)}} \cdot \det B \cdot \cancel{\det(P)}$. as $\det(A^{-1}) = \dfrac{1}{\det A}$ when $\det A \neq 0$

Theorem If $A \sim B$, then they have the same characteristic polynomial, and hence the same eigenvalues.

Given $B = P^{-1}AP$

$B - \lambda I = \underbrace{P^{-1}AP}_{B} - \lambda \underbrace{P^{-1}P}_{I} = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$. as $A(B+C) = AB + AC$

Hence $(A - \lambda I) \sim (B - \lambda I)$.

So $\det(B - \lambda I) = \det(A - \lambda I)$ (as shown above)

EROs and eigenvalues

We have seen previously that a replacement ERO does not change the determinant. Thus, if $A \xrightarrow{R_i + kR_j} B$, then $\det B = \det A$.

How does EROs affect eigenvalues?

Pg 280

In Exercises 21 and 22, A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

22. d. A row replacement operation on A does not change the eigenvalues.

False!

Consider the following example.

Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. $\det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = \lambda^2 - 2\lambda + 1$

The eigenvalue is 1 with multiplicity 2.

Let $A \xrightarrow{R_1 + R_2} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = A'$ $\quad \det(A' - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - 1 \times 1$

$$= \lambda^2 - 3\lambda + 2 - 1$$
$$= \lambda^2 - 3\lambda + 1$$

The eigenvalues are different!

$\lambda = \dfrac{3 \pm \sqrt{5}}{2}$ are the two eigenvalues of A'.

$\left(\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ are the solutions to } ax^2 + bx + c = 0 \right)$

Brief Matlab session on the project

The commands in Matlab that you will need are **eig**, **diag**, **sum**, **abs**, and may be **sort**.

If the real eigenvalue with the largest absolute value is < 1 (in absolute value), then the population will become extinct eventually.

the population will eventually die off here

```
%% Octave session from Lecture 28 on Thursday, Nov 21, 2013
%% Notice that there might be minor differences between Octave and
%% Matlab, but the results of your calculations should be the same.

octave:2> A = [0 0 .33; .18  0 0 ; 0 .71 .94]
A =
   0.00000   0.00000   0.33000
   0.18000   0.00000   0.00000
   0.00000   0.71000   0.94000

% Use the function eig to find the eigenvalues and the corresponding eigenvectors

octave:3> help eig

`eig' is a function from the file /usr/lib/octave/3.6.2/oct/i686-pc-cygwin/eig.oct
 -- Loadable Function: LAMBDA = eig (A)
 -- Loadable Function: LAMBDA = eig (A, B)
 -- Loadable Function: [V, LAMBDA] = eig (A)
 -- Loadable Function: [V, LAMBDA] = eig (A, B)
     Compute the eigenvalues and eigenvectors of a matrix.

     Eigenvalues are computed in a several step process which begins
     with a Hessenberg decomposition, followed by a
     Schur decomposition, from which the eigenvalues are apparent.  The
     eigenvectors, when desired, are computed by further manipulations
     of the Schur decomposition.

     The eigenvalues returned by `eig' are not ordered.

     See also: eigs, svd

Additional help for built-in functions and operators is
available in the on-line version of the manual.  Use the command
`doc <topic>' to search the manual index.


octave:4> [V,L]=eig(A)

V =
   0.68209 + 0.00000i   0.68209 - 0.00000i   0.31754 + 0.00000i
  -0.06241 - 0.58963i  -0.06241 + 0.58963i   0.05811 + 0.00000i
  -0.04505 + 0.42562i  -0.04505 - 0.42562i   0.94646 + 0.00000i
L =
Diagonal Matrix
  -0.02180 + 0.20592i                    0                    0
                    0   -0.02180 - 0.20592i                    0
                    0                    0    0.98359 + 0.00000i


% We could extract the eigenvalues from the diagonal of L using the command diag

octave:5> Lambdas = diag(L)
Lambdas =
  -0.02180 + 0.20592i
  -0.02180 - 0.20592i
   0.98359 + 0.00000i

% The absolute values of the eigenvalues are obtained using the
% function abs

octave:6> abs(Lambdas )
ans =
   0.20707
   0.20707
   0.98359
```