MATH 220 – Lecture 27 (11/19/2013)

Homework 5.1 due Friday Nov 22
5.2 due Tuesday Dec 3.

Recall If $\bar{x} \neq \bar{0}$ such that $A\bar{x} = \lambda \bar{x}$ for some λ, then $\bar{x}$ is an eigenvector of A corresponding to the eigenvalue λ.

$\lambda = 0$ is okay

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5. Is $\bar{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} -4 & 3 & 3 \\ 2 & -3 & -2 \\ -1 & 0 & -2 \end{bmatrix}$? If so, find the eigenvalue.

Just check if $A\bar{x} = \lambda\bar{x}$ for some λ.

$$A\bar{x} = \begin{bmatrix} -4 & 3 & 3 \\ 2 & -3 & -2 \\ -1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -15 \\ 10 \\ -5 \end{bmatrix} = -5\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} = -5\bar{x}.$$

So $\bar{x}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -5$.

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7. Is $\lambda = 4$ an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? If so, find one corresponding eigenvector.

$\lambda = 4$ is an eigenvalue if $(A-\lambda I)\bar{x} = \bar{0}$ has nontrivial solutions.

$$A-\lambda I = \begin{bmatrix} 3-\lambda & 0 & -1 \\ 2 & 3-\lambda & 1 \\ -3 & 4 & 5-\lambda \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \xrightarrow[R_3+3R_1]{-R_1,\text{ then } R_2-2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow[\text{then } -R_2]{R_3+4R_2}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad x_3 \text{ free} \qquad \bar{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} s, \quad s \in \mathbb{R}.$$

So $\lambda = 4$ is an eigenvalue. And $\bar{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector.

Def The **eigenspace** of A corresponding to the eigenvalue λ is the null space of $A-\lambda I$.

Notice that the eigenspace is indeed a subspace, i.e., it contains the zero vector. Hence the eigenspace of A corresponding to the eigenvalue λ consists of the zero vector along with all corresponding eigen vectors.

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In Exercises 9–16, find a basis for the eigenspace corresponding to each listed eigenvalue.

14. $A = \begin{bmatrix} 4 & 0 & -1 \\ 3 & 0 & 3 \\ 2 & -2 & 5 \end{bmatrix}, \lambda = 3.$

$$A - \lambda I = \begin{bmatrix} 4-\lambda & 0 & -1 \\ 3 & 0-\lambda & 3 \\ 2 & -2 & 5-\lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -3 & 3 \\ 2 & -2 & 2 \end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-3R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 6 \\ 0 & -2 & 4 \end{bmatrix} \xrightarrow[\text{then } R_3+2R_2]{R_2 \times -\frac{1}{3}}$$

x_3 free

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} s, \; s \in \mathbb{R}.$$

Hence a basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

We now consider certain special cases where we could guess some eigenvalues and eigenvectors more directly.

Eigenvalues of triangular matrices

Recall that if A is a triangular matrix, then $\det A$ is the product of the diagonal entries.

$* \longrightarrow$ zero or nonzero

If $A = \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix}$, $\det A = abc$.

Similarly, $\det(A - \lambda I) = \det\left(\begin{bmatrix} a-\lambda & * & * \\ 0 & b-\lambda & * \\ 0 & 0 & c-\lambda \end{bmatrix} \right) = (a-\lambda)(b-\lambda)(c-\lambda).$

To find eigenvalues we solve $\det(A-\lambda I) = 0$, i.e.,

$$(a-\lambda)(b-\lambda)(c-\lambda) = 0.$$

Hence $\lambda = a, b, c$ are the eigenvalues of A.

Theorem The eigenvalues of a triangular matrix are the entries on its diagonal.

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Find the eigenvalues of the matrices in Exercises 17 and 18.

18. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$ The matrix is lower triangular. Hence $\lambda = 5, 0, 3$ are its eigenvalues.

Other "easy" cases

20. Without calculation, find one eigenvalue and two linearly independent eigenvectors of $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Justify your answer.

Notice that all rows add up to the same number 6.

Adding the entries in all rows simultaneously $\equiv A\bar{x}$ for $\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

If $A = [\bar{a}_1\ \bar{a}_2\ \bar{a}_3]$, $A\bar{x} = \bar{a}_1 + \bar{a}_2 + \bar{a}_3$.

Check $\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $A\bar{x} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

So, $\lambda = 6$ is an eigenvalue, and $\bar{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector. But we want two LI eigenvectors!

What about $\cancel{\bar{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$? $A\bar{u} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for any λ.

does not work ☹!

Columns of A are not LI. So $A\bar{x}=\bar{0}$ has nontrivial solutions. Hence $(A-\lambda I)\bar{x}=\bar{0}$ has nontrivial solutions when $\lambda=0$.

$\bar{a}_1-\bar{a}_2=\bar{0}$ $\qquad$ $\bar{a}_2-\bar{a}_3=\bar{0}$

In fact, $\bar{u}_1=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $\bar{u}_2=\begin{bmatrix}0\\1\\-1\end{bmatrix}$ are both nontrivial solutions to $A\bar{x}=\bar{0}$, and hence for $(A-\lambda I)\bar{x}=\bar{0}$ for $\lambda=\bar{0}$. So, $\bar{u}_1, \bar{u}_2$ are LI eigenvectors corresponding to the eigenvalue $\lambda=0$.

Notice that $\bar{x}=\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\bar{u}_1=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ are indeed LI.
But they correspond to two different eigenvalues, $\lambda=6$ and $\lambda=0$, respectively.

On the other hand, $\bar{u}_1=\begin{bmatrix}1\\-1\\0\end{bmatrix}$ and $\bar{u}_2=\begin{bmatrix}0\\1\\-1\end{bmatrix}$ are two LI eigenvectors corresponding to the same eigenvalue $\lambda=0$.

Similarly, $\bar{u}_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}$ is another eigenvector corresponding to the eigenvalue $\lambda=0$. In fact, the set $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is LI.

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30. Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint:* Use Exercises 27 and 29.]

27. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. [*Hint:* Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]

29. Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint:* Find an eigenvector.]

First, we show that the eigenvalues of A and A^T are the same (Prob 27).

$\det(A-\lambda I)=0$ if λ is an eigenvalue of A

So, $\det\left[(A-\lambda I)^T\right] = \det(A-\lambda I) = 0$ as $\det A^T = \det A$

hence, $\det\left(A^T-(\lambda I)^T\right) = \det(A^T-\lambda I) = 0$ as $(A+B)^T = A^T+B^T$, and $I^T = I$

So λ is an eigenvalue of A^T.

Then we prove the row sum eigenvalue property (Prob 29).

Notice that finding the sum of all rows of A is equivalent to multiplying it by the vector of ones.

With $A = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]$ and $\bar{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \Big\updownarrow n$, we have

$$A\bar{u} = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_n]\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \bar{a}_1 + \bar{a}_2 + \cdots + \bar{a}_n = \begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix} \text{ here.}$$

all rows add up to s.

Hence we have $A\bar{u} = \begin{bmatrix} s \\ s \\ \vdots \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = s\bar{u}$.

In other words, $\lambda = s$ is an eigenvalue of A, and $\bar{u} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ is a corresponding eigenvector.

Now, notice that if the columns of A all add up to s, the rows of A^T all add up to s. Hence we can combine the two results above (Problems 27 and 29) to conclude that $\lambda = s$ is an eigenvalue of A.