# MATH 220 – Lecture 26 (11/14/2013)

Properties of determinants: $\det(AB) = \det(A)\cdot\det(B)$

$\det(A^T) = \det(A)$

Pg 176.

**31.** Show that if $A$ is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$.

diagonal matrices are both upper and lower triangular matrices at the same time!

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

diagonal matrix has all nondiagonal entries $= 0$.

We have $AA^{-1} = I$

Taking determinants on both sides,

$$\det(AA^{-1}) = \det I$$

$$= 1$$

product of $n$ copies of 1 on the diagonal

So $\det(A)\cdot\det(A^{-1}) = 1$

Hence, when $A$ is invertible, $\det(A) \neq 0$, and so

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Pg 176.

**36.** Suppose that $A$ is a square matrix such that $\det A^4 = 0$. Explain why $A$ cannot be invertible.

$A \cdot A \cdot A \cdot A$

$\det A^4 = (\det A)^4$ follows from $\det(AB) = \det A \cdot \det B$

Hence if $\det A^4 = 0$, $(\det A)^4 = 0$, i.e., $\det A = 0$.

Hence $A$ is not invertible.

Pg 176 (26 2)

**40.** Let $A$ and $B$ be $4 \times 4$ matrices, with $\det A = -1$ and $\det B = 2$. Compute:

a. $\det AB$  b. $\det B^5$  c. $\det 2A$

d. $\det A^TA$  e. $\det B^{-1}AB$

a. $\det AB = \det A \cdot \det B = -1 \times 2 = -2.$

b. $\det B^5 = (\det B)^5 = (2)^5 = 32.$

c. $\det(2A) = (2)^4 \cdot \det A = 16 \cdot -1 = -16$

every row of $A$ is scaled by 2, and there are 4 rows

For $A \in \mathbb{R}^{n\times n}$, $\det(cA) = c^n \det(A)$.

n-rows getting scaled by c each.

d. $\det(A^TA) = \det(A^T) \cdot \det A = \det(A) \cdot \det(A) = (-1)^2 = 1.$

e. $\det(B^{-1}AB) = \det(B^{-1}) \cdot \det(A) \cdot \det(B)$

$= \dfrac{1}{\cancel{\det(B)}} \cdot \det(A) \cdot \cancel{\det(B)}$ as $\det(B) = 2 \neq 0$

$= -1.$

Also, $\det(B^{-1}AB) = \det A = -1$

Similarly, $\det(A^{-1}BA^{-1}) = \det(A^{-1}) \cdot \det(B) \cdot \det(A^{-1})$

$= \dfrac{1}{\det A} \cdot \det B \cdot \dfrac{1}{\det A} = -1 \cdot 2 \cdot -1$

$= 2.$

# Eigenvalues and eigenvectors (Chapter 5)

**Motivation** Given $A \in \mathbb{R}^{n\times n}$, can we say something more about the images of the LT $\bar{x} \mapsto A\bar{x}$, apart from the basis for Col A?

In particular, are there vectors $\bar{x}$ whose images under the LT "look very much like" $\bar{x}$?
More precisely, the images are just scaled versions of $\bar{x}$.

The zero vector always fits this criterion, but we are interested in non-trivial vectors.

**Def** $\bar{x} \in \mathbb{R}^n$ is an **eigenvector** of $A \in \mathbb{R}^{n\times n}$ if $\bar{x}$ is nonzero, and for some scalar $\lambda$, we have $\mathbf{A\bar{x} = \lambda\bar{x}}$.
In this case, $\lambda$ is an **eigenvalue** of $A$, and $\bar{x}$ is the eigenvector corresponding to $\lambda$.

at least one entry is $\neq 0$.

## Some 2x2 examples

(a) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Matrix of the geometric LT that reflects points across the $y=x$ line.

(flips x- and y-coordinates)

Need to find $\bar{x}$ and $\lambda$ such that $A\bar{x}=\lambda\bar{x}$, and $\bar{x}\neq\bar{0}$.

$$A\bar{x}=\lambda\bar{x}$$

$\Rightarrow A\bar{x}-\lambda\bar{x}=\bar{0}$ or $A\bar{x}-\lambda I\bar{x}=\bar{0}$ where $I$ is the 2x2 identity matrix

$\Rightarrow \underbrace{(A-\lambda I)}_{2\times 2 \text{ matrix}}\bar{x}=\bar{0}$ two unknowns – $\bar{x}$ and $\lambda$.

We want $A-\lambda I$ to be not invertible, as we are looking for nontrivial solutions to $(A-\lambda I)\bar{x}=\bar{0}$.

So $\det(A-\lambda I)=0$ only one unknown ($\lambda$)

$$A-\lambda I=\begin{bmatrix}0&1\\1&0\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}0-\lambda&1\\1&0-\lambda\end{bmatrix}=\begin{bmatrix}-\lambda&1\\1&-\lambda\end{bmatrix}$$

$\det(A-\lambda I)=(-\lambda)^2-1=\lambda^2-1=0$ when $\lambda=\pm1$.

Hence, there are two eigenvalues to $A$, $\lambda_1=1$, $\lambda_2=-1$.

To find an eigenvector corresponding to $\lambda_1=1$, we find a non-trivial solution to $(A-\lambda_1 I)\bar{x}=\bar{0}$.

$$A-\lambda_1 I=\begin{bmatrix}-1&1\\1&-1\end{bmatrix}\xrightarrow[\text{then } -R_1]{R_2+R_1}\begin{bmatrix}1&-1\\0&0\end{bmatrix}$$

$x_2$ free. $x_1-x_2=0$, i.e., $x_1=x_2$

$\Rightarrow \bar{x}=\begin{bmatrix}1\\1\end{bmatrix}s$, $s\in\mathbb{R}$.

So $\bar{x}=\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda=1$.

(b) $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\qquad A - \lambda I = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$

$(3-\lambda)\times(3-\lambda) - 1\times 1$

$\det(A - \lambda I) = 0 \Rightarrow (3-\lambda)^2 - 1 = 0$

$3+3$ = sum of diagonal entries

$\Rightarrow \lambda^2 - 6\lambda + 8 = 0$

$\det A = 3\cdot 3 - 1\cdot 1 = 8$ $\qquad (\lambda-2)(\lambda-4) = 0$ $\qquad \det A = 3\times 3 - 1\times 1$

$ad - bc$

There are two eigenvalues, $\lambda = 2$, $\lambda = 4$.

In general, for $A \in \mathbb{R}^{2\times 2}$, we have

$$\det(A - \lambda I) = \lambda^2 - (\text{trace}(A)) + \det A.$$

**Def** $\text{trace}(A)$ = sum of diagonal entries, when $A \in \mathbb{R}^{n\times n}$.

We can find an eigenvector corresponding to the eigenvalue $\lambda = 4$, for instance, just as in the previous example.

With $\lambda = 4$, $A - \lambda I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow[\text{then } R_1\times(-1)]{R_2 + R_1} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $\qquad \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} s$, $s \in \mathbb{R}$.

Thus, $\bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ corresponding to the eigenvalue $\lambda = 4$.

(c) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates vectors $90^\circ$ CCW about the origin

$\det(A - \lambda I) = 0$ $\quad [A - \lambda I] = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$

$\Rightarrow (-\lambda)^2 + 1 = 0$ or $\lambda^2 + 1 = 0$.

No real eigenvalues ☹!

Result If $A$ is symmetric, i.e., $A_{ij} = A_{ji}$, or $A^T = A$, then $A$ has only real eigenvalues.

If $A$ is antisymmetric, i.e., $A_{ij} = -A_{ji}$, $A$ has only non-real eigenvalues.

In Math 220, we will typically concern ourselves with real eigenvalues.