MATH 220 – Lecture 20 (10/24/2013)

Next week: Tuesday: Prof McDonald
Thursday: No class

Subspaces (Section 2.8)

Motivation: $\mathbb{R}^2$ is like an infinite sheet of paper sitting on a flat "table", so to say. Now imagine the same infinite sheet sitting in 3D space. When can we guarantee that the desired properties of $\mathbb{R}^2$ are all retained by this sheet now sitting in $\mathbb{R}^3$?

Def H is a **subspace** of $\mathbb{R}^n$ if

1. the zero vector is in H;
2. for $\bar{u}, \bar{v}$ in H, $\bar{u}+\bar{v}$ is also in H; and
3. for $\bar{u}$ in H and scalar c, $c\bar{u}$ is also in H.

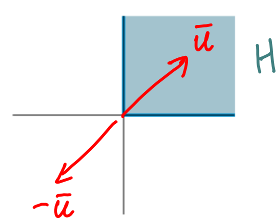
H is closed under vector addition and scalar multiplication.

Note: $\mathbb{R}^n$ as well as $\{\bar{0}\}$ (zero vector) are both subspaces of $\mathbb{R}^n$.

Prob 2, pg 151

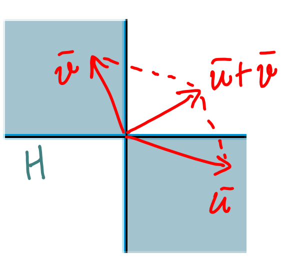
Exercises 1–4 display sets in $\mathbb{R}^2$. Assume the sets include the bounding lines. In each case, give a specific reason why the set H is *not* a subspace of $\mathbb{R}^2$. (For instance, find two vectors in H whose sum is *not* in H, or find a vector in H with a scalar multiple that is not in H. Draw a picture.)

1.



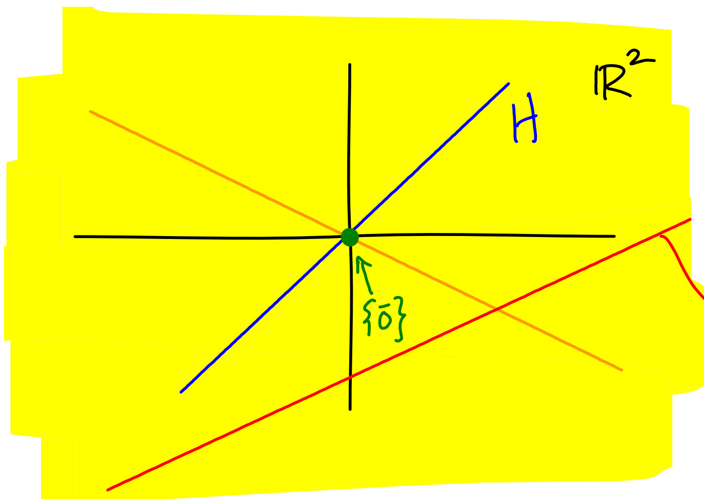
$c\bar{u}$ is not in H for any $c<0$. So H is not a subspace

2.



$\bar{u}+\bar{v}$ is not in H.
So H is not a subspace.

What are (valid) subspaces of $\mathbb{R}^2$?



$\mathbb{R}^2$, $\{\bar{0}\}$ are subspaces.

So are lines passing through the origin.

not a subspace, as origin is not included.

Def Span $(\{\bar{v}_1, \ldots, \bar{v}_p\})$ is a subspace of $\mathbb{R}^m$, when each $\bar{v}_j \in \mathbb{R}^m$ (the set of all linear combinations of $\bar{v}_1, \ldots, \bar{v}_p$). We call this subspace the **subspace generated by, or spanned by** $\bar{v}_1, \ldots, \bar{v}_p$.

Prob 5, pg 151

5. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} -3 \\ -3 \\ 10 \end{bmatrix}$. Determine if $\mathbf{w}$ is in the subspace of $\mathbb{R}^3$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$A = \begin{array}{c} \begin{array}{cc} \bar{v}_1 & \bar{v}_2 \end{array} \\ \begin{bmatrix} 1 & -2 \\ 3 & -3 \\ -4 & 7 \end{bmatrix} \end{array}$$

The question asks "is $\bar{w}$ in Span$(\{\bar{v}_1, \bar{v}_2\})$"?

Reword: With $A = [\bar{v}_1\ \bar{v}_2]$, is $A\bar{x} = \bar{w}$ consistent?

Notice that we need **not** solve for $\bar{x}$ – we just need to determine if the system is consistent or not in order to answer the question.

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -3 \\ -4 & 7 \end{bmatrix} \quad \bar{w} = \begin{bmatrix} -3 \\ -3 \\ 10 \end{bmatrix}$$

$$[A|\bar{w}] = \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 3 & -3 & -3 \\ -4 & 7 & 10 \end{array}\right] \xrightarrow[R_3+4R_1]{R_2-3R_1} \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 3 & 6 \\ 0 & -1 & -2 \end{array}\right] \xrightarrow{R_3+\frac{1}{3}R_2} \left[\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

system is consistent. So, $\bar{w} \in \text{span}(\{\bar{v}_1, \bar{v}_2\})$.

We will now study two subspaces related to a matrix A, and their relationships to various concepts we have already seen — consistency of $A\bar{x}=\bar{b}$, one-to-one and onto LTs defined by A, etc.

Column Space and Null Space of $A \in \mathbb{R}^{m \times n}$

Def The **column space of A** is the set of all linear combinations of the columns of A. We denote it $\text{Col}\,A$.

$\text{Col}\,A$ is a subspace of $\mathbb{R}^m$.

$\bar{b} \in \mathbb{R}^m$ is in $\text{Col}\,A$ if $A\bar{x}=\bar{b}$ is consistent.

The **null space of A** is the set of all solutions to $A\bar{x}=\bar{0}$. We denote it by $\text{Nul}\,A$.

Since any $\bar{x}$ that is a solution to $A\bar{x}=\bar{0}$ is in $\mathbb{R}^n$,

$\text{Nul}\,A$ is a subspace of $\mathbb{R}^n$.

Since $\text{Col}\,A$ is the set of all linear combinations of the columns of A, it satisfies the definition of a subspace in a straightforward manner.

Let us check the definition for Nul A being a subspace now.

1. $A\bar{0} = \bar{0}$ (trivial solution). So $\bar{0} \in \text{Nul}\, A$.
2. For $\bar{x}_1, \bar{x}_2$ such that $A\bar{x}_1 = \bar{0}$ and $A\bar{x}_2 = \bar{0}$, indeed

   $\bar{x}_1, \bar{x}_2 \in \text{Nul}\, A$

   $A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = \bar{0} + \bar{0} = \bar{0}$.

   So $\bar{x}_1 + \bar{x}_2$ is also in Nul A.
3. For $\bar{x}$ in Nul A, i.e., $A\bar{x} = \bar{0}$, consider $A(c\bar{x})$.

   $A(c\bar{x}) = c(A\bar{x}) = c\bar{0} = \bar{0}$. So $c\bar{x} \in \text{Nul}\, A$.

Prob 7, Pg 157

7. Let

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix},$$

$$\mathbf{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}, \quad \text{and} \quad A = [\, \mathbf{v}_1 \; \mathbf{v}_2 \; \mathbf{v}_3 \,].$$

a. How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in Col A?

c. Is $\mathbf{p}$ in Col A? Why or why not?

$$[A \mid \bar{p}] = \left[\begin{array}{ccc|c} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{array}\right]$$

(a). Three. $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is just a collection of the three vectors.

(b). Infinitely many. Remember, Col A is the set of all linear combinations of the columns of A.

(c) $\bar{p} \in \text{Col}\, A$ if $A\bar{x} = \bar{p}$ is consistent.

$$[A|\bar{p}] = \left[\begin{array}{ccc|c} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2+4R_1} \left[\begin{array}{ccc|c} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 2 & 5 & -7 \end{array}\right] \xrightarrow{R_3+\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

System is consistent. So $\bar{p} \in \text{Col } A$.

Again, notice that we need not solve the system $A\bar{x}=\bar{p}$.

<u>Prob 9, Pg 151</u>

9. With A and $\mathbf{p}$ as in Exercise 7, determine if $\mathbf{p}$ is in Nul A.

$\bar{p} \in \text{Nul } A$ if $A\bar{p} = \bar{0}$.

$$A\bar{p} = \begin{bmatrix} 2 & -3 & -4 \\ -8 & 8 & 6 \\ 6 & -7 & -7 \end{bmatrix}\begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix} = \begin{bmatrix} 2\times 6 + (-3)\times(-10) + (-4)\times 11 \\ -8\times 6 + 8\times(-10) + 6\times 11 \\ 6\times 6 + (-7)\times(-10) + (-7)\times 11 \end{bmatrix} = \begin{bmatrix} -2 \\ -62 \\ 29 \end{bmatrix} \neq \bar{0}.$$

So $\bar{p} \notin \text{Nul } A$.

"not an element of".

<u>Basis for a subspace H</u>

We saw that a subspace, by definition, has infinitely many vectors. Hence we try to work with a finite subset of these vectors which generates the entire subspace. It also makes sense to study such a finite set that is also minimal, i.e., has the smallest number of vectors. It turns out that such a minimal set is LI.

Def A linearly independent set in H which spans H is a **basis** for H.

Equivalently, a basis is a minimal subset of H that generates H.

Example The unit vectors $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ form a basis for $\mathbb{R}^3$.

Notice that the set $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is LI, and any $\bar{x} \in \mathbb{R}^3$ can be written as a unique linear combination of $\bar{e}_1, \bar{e}_2$, and $\bar{e}_3$.

In general, $\{\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n\}$ forms a basis for $\mathbb{R}^n$, where $\bar{e}_j$ is the j^{th} unit n-vector. This is the standard basis for $\mathbb{R}^n$.

In the next lecture, we will talk about bases for Col A and Nul A.

plural of basis