MATH 220 – Lecture 18 (10/17/2013)

We saw:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A^{-1} exists when $\det(A) = ad - bc \neq 0$,

and in that case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Inverting n×n matrices

We want to find B such that $AB = I$.

Theorem A is invertible if and only if it is row equivalent to I, the identity matrix. In other words, we can transform A to I using a sequence of EROs.

The same sequence of EROs will convert I to A^{-1}!

So $\quad [A \; I] \xrightarrow[\text{to reduced echelon form}]{\text{EROs}} [\underline{I} \; A^{-1}]$

If we do not get I in place of A, A is not invertible

If we do not get I in place of A, it means A is not row equivalent to I. Hence A is not invertible.

Prob 31, pg 110

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}.$$ Find A^{-1} if it exists.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 + 3R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 + 3R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \xrightarrow[\text{and then } R_3 \times \frac{1}{2}]{R_1 + R_3,\ R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

So, A is invertible, and $A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$.

Check: $$AA^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 1 & \boxed{0} \\ 0 & 0 & 1 \end{bmatrix}$$

$-3 \times 1 + 1 \times 1 + 4 \times \frac{1}{2} = 0$

$1 \times 8 + 0 \times 10 + (-2) \times \frac{7}{2} = 8 - 7 = 1$

Of course, you need not check the result in each case. We just did it here for the sake of completeness.

We now look at a few "proof"-type problems, where we will use the many properties of matrix multiplication and inverses that we have discussed so far.

Recall: If there exists B such that $AB = BA = I$, then $B = A^{-1}$, $A = B^{-1}$, A and B are both invertible.

Prob 14, pg 110

14. Suppose $(B-C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$.

$(B-C)D = 0$ → $m \times n$ zero matrix

$B, C \in \mathbb{R}^{m\times n}$, D is invertible.

Notice that D is $n \times n$, for the product $(B-C)D$ to be defined for a square matrix D.

$$\big((B-C)D = 0\big)D^{-1}$$ multiply on the right by D^{-1} (as D^{-1} exists)

$$(B-C)\underbrace{DD^{-1}} = 0 \cdot D^{-1} = 0$$ $(DD^{-1} = I)$

$$(B-C)I = 0$$

So, $B - C = 0$, hence $B = C$. (as $AI = A$ for any matrix A)

Prob 20, pg 110

20. Suppose A, B, and X are $n \times n$ matrices with A, X, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B \qquad (3)$$

a. Explain why B is invertible.

b. Solve equation (3) for X. If a matrix needs to be inverted, explain why that matrix is invertible.

(a). We try to find B^{-1}. By explicitly giving the expression for B^{-1}, we can demonstrate that B is invertible.

$$X\left((A-AX)^{-1} = X^{-1}B\right)$$ multiply on the left by X

$$X(A-AX)^{-1} = \underbrace{X X^{-1}}B$$ (multiplication is associative)

$$= I\, B = B$$ $(XX^{-1} = I)$

So, $B^{-1} = \left[X(A-AX)^{-1}\right]^{-1}$

$$= \left[(A-AX)^{-1}\right]^{-1} X^{-1}$$ (as $(AB)^{-1} = B^{-1}A^{-1}$)

$$= (A-AX) \cdot X^{-1}$$ (as $(A^{-1})^{-1} = A$)

Since B^{-1} exists, B is invertible.

(b) To solve $(A-AX)^{-1} = X^{-1}B$ for X, we "isolate" X.

Taking inverse on both sides,

$$[(A-AX)^{-1}]^{-1} = (X^{-1}B)^{-1}$$

$$\xrightarrow{\text{"implies"}} \quad A - AX = B^{-1}(X^{-1})^{-1} \qquad (\text{as } (A^{-1})^{-1} = A \text{ and } (AB)^{-1} = B^{-1}A^{-1})$$

$$= B^{-1}X$$

$$\Rightarrow \quad A = AX + B^{-1}X = (A+B^{-1})X \qquad (\text{as } A(B+C) = AB + AC)$$

Hence we have $(A+B^{-1})X = A$

Notice that since A is invertible, we must have that $(A+B^{-1})$ is invertible (since $A^{-1} = X^{-1}[(A+B^{-1})]^{-1}$).

$$\xrightarrow{\text{"implies"}} \quad \underbrace{[(A+B^{-1})]^{-1}\{(A+B^{-1})X}\; = A\} \qquad (\text{As } A(BC) = (AB)C)$$

$$\underbrace{(A+B^{-1})^{-1}(A+B^{-1})}_{I}X = (A+B^{-1})^{-1}A$$

$$\boxed{X = (A+B^{-1})^{-1}A}$$

Properties of invertible matrices (Section 2.3)

We now connect the various properties of systems of equations $A\bar{x}=\bar{b}$, linear transformations, and inverses of matrices. It turns out that many of the properties are equivalent.

Invertible matrix theorem (IMT)

For an $n\times n$ matrix A, the following statements are equivalent.

(a) A is invertible.

(b) A is row equivalent to I_n.

(c) A has n pivot positions.

(d) $A\bar{x}=\bar{0}$ has only trivial solution.

(e) Columns of A are LI.

(f) The LT $\bar{x}\mapsto A\bar{x}$ is one-to-one

(h) Columns of A span $\mathbb{R}^n$.

⋮

(more statements to follow in the next lecture...)