MATH 220 – Lecture 17 (10/15/2013)

Midterm: Scores for midterm will be curved

Offer:

* If you score 90+/100 in the final, midterm will be replaced by final
* If you score 85-89.9/100, the percentages will be reset as
  midterm - 15% final - 50%

Inverse of a matrix (Section 2.2)

For nonzero numbers, we have the concept of multiplicative inverse, e.g.,

$$(5)^{-1} = \frac{1}{5},$$

with the property that $5 \cdot (5)^{-1} = 5 \cdot \frac{1}{5} = 1.$

We define the analogous concept for matrices.

Def If $AB = I$ and $BA = I$, then B is called the **inverse** of A, and A the inverse of B. We denote this fact by $(A)^{-1} = B$, $B^{-1} = A$.

I is the identity matrix. For both AB and BA to be defined, A, B must be square matrices, i.e., $n \times n$ matrices.

Def If the inverse of A exists, then A is **invertible**.

e.g., $A = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 2\times 4 + 1\times(-7) & 2\times(-1) + 1\times 2 \\ 7\times 4 + 4\times(-7) & 7\times(-1) + 4\times 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So $B = A^{-1}$ and $A = B^{-1}$!

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In general, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A^{-1} exists if $\mathbf{ad - bc \neq 0}$, and in this case

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

swap diagonal entries, and change sign of off-diagonal entries

The quantity $ad-bc$ is called the **determinant** of the matrix A, denoted as $\det(A)$.

A 2×2 matrix A is invertible if and only if $\det(A) \neq 0$.

↳ extends to $n\times n$ matrices in general

Check! $AA^{-1} = I$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad-bc \neq 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{(ad-bc)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(ad-bc)} \begin{bmatrix} ad+b(-c) & -ab+ba \\ cd+d(-c) & c(-b)+da \end{bmatrix}$$

$$= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

e.g., $B = \begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$ $\quad \det(B) = 4\times 2 - (-1)\times(-7) = 1 \neq 0$, so B^{-1} exists.

$$B^{-1} = \frac{1}{\det(B)} \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix} \;(=A)$$

Why study inverses?

If x is a scalar, and we have the equation $5x=3$, we could solve for x by multiplying the equation by the inverse of 5:

$$\frac{1}{5}(5x=3) \Rightarrow \frac{1}{5}\cdot 5x = \frac{1}{5}3 \quad \text{i.e., } x = 3/5.$$

We can extend this result to matrices as follows.

For $A\bar{x}=\bar{b}$ with $A \in \mathbb{R}^{n\times n}$, if A^{-1} exists, then the system has a unique solution for every $\bar{b} \in \mathbb{R}^n$ given as $\bar{x} = A^{-1}\bar{b}$.

Prob 6 pg 109

$$7x_1+3x_2=-9$$
$$-6x_1-3x_2=4$$

Solve the system using inverses.

$$A\bar{x}=\bar{b} \text{ with } A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}, \bar{b} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

$$\det A = 7\times(-3) - (-6)\times 3 = -3. \quad \text{So } A^{-1} \text{ exists.}$$

$$A^{-1} = \frac{1}{-3}\begin{bmatrix} -3 & -3 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix}. \quad \bar{x} = A^{-1}\bar{b} = \begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix}\begin{bmatrix} -9 \\ 4 \end{bmatrix} = \begin{bmatrix} 1\times -9 + 1\times 4 \\ -2\times -9 + -7/3\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} -5 \\ 26/3 \end{bmatrix}.$$

Properties of matrix inverses

1. $(A^{-1})^{-1} = A$
2. $(AB)^{-1} = B^{-1}A^{-1}$
3. $(A^T)^{-1} = (A^{-1})^T$

inverse of product = product of inverses in the reverse order

If $AB = I$, then $B = A^{-1}$

$$(AB)\cdot(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

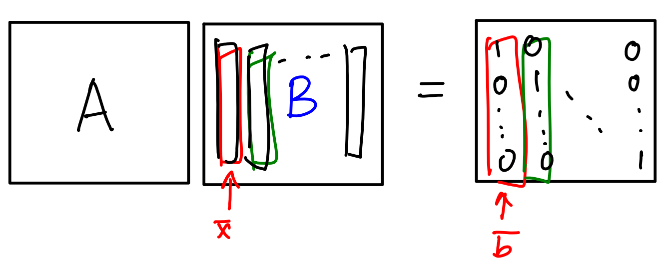
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

How to invert $n \times n$ matrices in general?

We need to find $B = A^{-1}$ such that

$$\mathbf{AB = I}$$

We know how to solve $A\bar{x} = \bar{b}$.



Collection of n systems all with same A matrix.

We form the big augmented matrix $[A \mid I]$, and reduce it to reduced row echelon form.