MATH 220 - Lecture 16 (10/10/2013)

This lecture was handled by Prof. Judi McDonald. A separate handout of her notes is also available on the course web page. What follows is the scribe for my own coverage of Section 2.1

Matrix Operations (Section 2.1)

$A_{m \times n}$ matrix is typically represented as follows.

$$
\begin{array}{c c c}
 & & \text{column } j \\
\text{Row } i & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} & = [A_{ij}]
\end{array}
$$

another notation for matrix A

Column representation:

$A = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_j \cdots \bar{a}_n]$ Each column is an m-vector.

Matrix addition

Let A and B be two matrices.

$C = A + B$ is defined only when both A & B are of the same size, say, $m \times n$. The sum C is also an $m \times n$ matrix.

Then $C_{ij} = A_{ij} + B_{ij}$

entry in Row-i & column-j of matrix C

<u>Scalar multiplication</u> Let r be a scalar.

If $C = rA$, then $C_{ij} = rA_{ij}$ for all i and j.
(i.e., multiply each entry by r).

<u>Prob 2, pg 116</u> $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$

$A + 2B = ?$ $\quad 2B = 2 \times \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 2\times 7 & 2\times -5 & 2\times 1 \\ 2\times 1 & 2\times -4 & 2\times -3 \end{bmatrix} = \begin{bmatrix} 14 & -10 & 2 \\ 2 & -8 & -6 \end{bmatrix}$

$$A + 2B = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 14 & -10 & 2 \\ 2 & -8 & -6 \end{bmatrix} = \begin{bmatrix} 16 & -10 & 1 \\ 6 & -13 & -4 \end{bmatrix}$$

<u>Properties of matrix addition and scalar multiplication</u>

A, B, C are $m \times n$ matrices, r, s are Scalars.

1. $A + B = B + A$
2. $(A+B) + C = A + (B+C)$
3. $A + 0 = 0 + A = A$
4. $r(A+B) = rA + rB$
5. $(r+s)A = rA + sA$
6. $r(sA) = (rs)A$

zero matrix : $m \times n$ matrix of all zeroes

Matrix Products

We have seen $A\bar{x} = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \bar{a}_1 x_1 + \bar{a}_2 x_2 + \cdots + \bar{a}_n x_n.$

Let A be $m \times n$ and B be $n \times p$ matrix.

$C = AB$? $B = [\bar{b}_1 \; \bar{b}_2 \cdots \bar{b}_p]$

We know how to find $A\bar{b}_j$ for $j = 1, 2, \ldots, p$. Then,

$C = [A\bar{b}_1 \; A\bar{b}_2 \cdots A\bar{b}_p]$, i.e., find each matrix-vector product $A\bar{b}_j$, stack these products, which are all m-vectors, together as columns to get AB.

> The product AB is defined only when the # columns in A is equal to the # rows in B.

Prob 2, pg 116 $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}_{2\times 3}$, $C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}_{2\times 2}$

The products CA and CB are defined, but AC (or BC) is NOT defined

In general, $AB \neq BA$. In fact, BA (or AB) may not even be defined!

$$CB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \left[7\begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad -5\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad 1\begin{bmatrix} 1 \\ -2 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}.$$

Row-Column definition of AB $A_{m\times n}$, $B_{n\times p}$

$$\text{Row-}i \rightarrow \underbrace{\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix}}_{A} \underbrace{\begin{array}{c} \text{Column } j \\ \downarrow \\ \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix} \end{array}}_{B} = \underbrace{\begin{bmatrix} & & \\ & c_{ij} & \\ & & \end{bmatrix}}_{C}$$

The product C will be an $m\times p$ matrix

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Properties of matrix multiplication

Assume all matrix products written here are defined.

1. $A(BC) = (AB)C$
2. $A(B+C) = AB + AC$
3. $(B+C)A = BA + CA$
4. $r(AB) = (rA)B = A(rB)$
5. $I_m A = A I_n = A$, where I_m is the $m\times m$ identity matrix

$A_{m\times n}$

$$I_m = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{bmatrix}_{m\times m}$$

Prob 10 pg 116

10. Let $A = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix}$. Verify that $AB = AC$ and yet $B \neq C$.

$$AB = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \boxed{3\times(-1) + (-6)\times 3} & 3\times 1 + (-6)\times 4 \\ (-1)\times(-1) + 2\times 3 & (-1)\times 1 + 2\times 4 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}.$$

$$AC = \begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -3 & -5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3\times(-3) + (-6)\times 2 & 3\times(-5) + (-6)\times 1 \\ (-1)\times(-3) + 2\times 2 & (-1)\times(-5) + 2\times 1 \end{bmatrix} = \begin{bmatrix} -21 & -21 \\ 7 & 7 \end{bmatrix}.$$

Hence $AB = AC$ here, even though $B \neq C$.

Transpose of a matrix

notation for transpose

Interchange rows and columns of matrix A to get A^T.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 4 & -2 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 4 \\ 1 & -2 \\ 0 & 6 \end{bmatrix}$$

2×3 $\qquad$ 3×2

If A is $m \times n$,
A^T is $n \times m$.

Properties of matrix transposes

a. $(A^T)^T = A$.

b. $(A+B)^T = A^T + B^T$

c. $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$ → transpose of a product = product of transposes in reverse order.

e.g., $A_{2\times 3}$, $B_{3\times 5}$. So AB is 2×5, and $(AB)^T$ is 5×2

A^T is 3×2, B^T is 5×3. Here $A^T B^T$ is not defined.

But $B^T A^T$ is 5×2.

Prob 27, pg 117

$\bar{u}$ and $\bar{v}$ are 3×1 matrices.

$$\bar{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}, \quad \bar{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

scalar

$$\bar{u}^T \bar{v} = \underset{1\times 3}{\begin{bmatrix} -2 & 3 & -4 \end{bmatrix}} \underset{3\times 1}{\begin{bmatrix} a \\ b \\ c \end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix} -2a+3b-4c \end{bmatrix}}$$

Scalar product of two vectors.

$$\underset{3\times 1}{\bar{u}} \, \underset{1\times 3}{\bar{v}^T} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \underset{3\times 3}{\begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}}$$

$$(\bar{u}\,\bar{v}^T)^T = (\bar{v}^T)^T(\bar{u})^T = \bar{v}\,\bar{u}^T \rightarrow \text{also a } 3\times3 \text{ matrix}$$

$$(\bar{u}^T\bar{v})^T = \bar{v}^T(\bar{u}^T)^T = \bar{v}^T\bar{u} \rightarrow \text{same scalar as } \bar{u}^T\bar{v} \ (1\times1)$$

(Similar to) Problem 17, pg 110

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}, \quad AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}. \quad \text{Find } B?$$

2×2 $\quad 2\times3$ $\quad$ AB is 2×3

B should be 2×3, so that AB is indeed 2×3.

Let $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

System 1: $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 5 & 6 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 4 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right] \quad \begin{array}{l} b_{11} = 7 \\ b_{21} = 4 \end{array}$$

System 2: $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$, $\quad b_{12} = -8$, $b_{22} = -5$

System 3: $\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $\quad b_{13} = 1$, $b_{23} = 1$

$$B = \begin{bmatrix} 7 & -8 & 1 \\ 4 & -5 & 1 \end{bmatrix}$$