# MATH 220 – Lecture 12 (09/26/2013)

Reminder: Midterm on Thursday Oct 3 in Todd 125.
Covers sections 1.1–1.9 (1.6 not included)

HW on Section 1.9 due on Tuesday, Oct 1.

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Recall: $T: \mathbb{R}^n \to \mathbb{R}^m$ is an LT, then $T(\bar{x}) = A\bar{x}$, where

$$A \in \mathbb{R}^{m\times n} \text{ with } A = [T(\bar{e}_1)\; T(\bar{e}_2) \cdots T(\bar{e}_n)].$$

Prob 22, pg 78

**22.** Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with $T(x_1, x_2) = (2x_1 - x_2, -3x_1 + x_2, 2x_1 - 3x_2)$. Find $\mathbf{x}$ such that $T(\mathbf{x}) = (0, -1, -4)$.

$$T(x_1,x_2) = (2x_1-x_2,\ -3x_1+x_2,\ 2x_1-3x_2).$$

essentially specifies the $A$ matrix such that $T(\bar{x}) = A\bar{x}$

$T(\bar{x}) = A\bar{x}$ where $A \in \mathbb{R}^{3\times 2}$ with $A = [T(\bar{e}_1)\; T(\bar{e}_2)]$.

$$\bar{e}_1 = \begin{bmatrix}1\\0\end{bmatrix},\ \bar{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}. \quad T(\bar{e}_1) = T(1,0) = \begin{bmatrix}2\times1-0\\-3\times1+0\\2\times1-0\end{bmatrix} = \begin{bmatrix}2\\-3\\2\end{bmatrix}.$$

could jump directly to A

$$T(\bar{e}_2) = T(0,1) = \begin{bmatrix}-1\\1\\-3\end{bmatrix}. \quad \text{So, } A = \begin{bmatrix}2 & -1\\-3 & 1\\2 & -3\end{bmatrix}.$$

Find $\bar{x}$ s.t. $T(\bar{x}) = (0,-1,-4)$.

Reword: Find a solution to $A\bar{x} = \bar{b}$, where $\bar{b} = \begin{bmatrix}0\\-1\\-4\end{bmatrix}$.

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ -3 & 1 & -1 \\ 2 & -3 & -4 \end{array}\right] \xrightarrow{R_1 \times \frac{1}{2}} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ -3 & 1 & -1 \\ 2 & -3 & -4 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+3R_1} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -1 \\ 0 & -2 & -4 \end{array}\right] \xrightarrow{R_2 \times -2} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{array}\right]$$

$$\xrightarrow[R_1+\frac{1}{2}R_2]{R_3+2R_2} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$\bar{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the unique vector such that $T(\bar{x}) = \bar{b}$ here.

(as there is a pivot in every column)

## Onto and 1-to-1 transformations

$T: \mathbb{R}^n \to \mathbb{R}^m \quad T(\bar{x}) = A\bar{x}, \quad A \in \mathbb{R}^{m \times n}.$

Recall

T is **onto** if every $\bar{b} \in \mathbb{R}^m$ has **at least** one $\bar{x} \in \mathbb{R}^n$ such that $T(\bar{x}) = \bar{b}$.

T is **1-to-1** if every $\bar{b} \in \mathbb{R}^m$ has **at most** one $\bar{x} \in \mathbb{R}^n$ such that $T(\bar{x}) = \bar{b}$.

Theorem 12

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if columns of A span $\mathbb{R}^m$, i.e., iff A has a pivot in every row.
   "if and only if"
2. T is one-to-one if and only if the columns of A are LI, i.e., iff A has a pivot in every column.

Probs 29, 30, pg 79

Describe all possible echelon forms.

29. $T: \mathbb{R}^3 \to \mathbb{R}^4$ is 1-to-1

We want all echelon forms of a $4 \times 3$ matrix with a pivot in every column.

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \end{bmatrix}$$ is the only possible echelon form.

30. $T: \mathbb{R}^4 \to \mathbb{R}^3$ is onto.

We are looking for echelon forms of a $3 \times 4$ matrix with a pivot in every row.

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

are the possible echelon forms.

Reminder

$\blacksquare \to$ any nonzero number

$* \to$ any number (zero or nonzero)

from Section 1.2

# Prob 24, pg 78-79 True/False

**24.** a. If $A$ is a $4 \times 3$ matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^3$ onto $\mathbb{R}^4$.

b. Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

c. The columns of the standard matrix for a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ are the images of the columns of the $n \times n$ identity matrix under $T$.

d. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

e. The standard matrix of a horizontal shear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the form $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, where $a$ and $d$ are $\pm 1$.

(a) False. A 4x3 matrix cannot have a pivot in every row.

(b) True $T: \mathbb{R}^n \to \mathbb{R}^m$ is an LT means $T(\bar{x}) = A\bar{x}$ where

$$A = [T(\bar{e}_1)\ T(\bar{e}_2) \dots T(\bar{e}_n)].$$ $e_j$ is the $j^{th}$ unit vector.

(c) True.

The nxn identity matrix is $\begin{array}{cc} \overset{\longleftarrow\; n \;\longrightarrow}{} & \\ \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & 1 \end{bmatrix} & \updownarrow n \end{array}$, which

is $[\bar{e}_1\ \bar{e}_2 \dots \bar{e}_n]$.

(d) False. The definition given is satisfied by **any** transformation, i.e., by any function (or map). For a 1-to-1 mapping, we need that for every $\bar{b} \in \mathbb{R}^m$, there must exist at most one $\bar{x} \in \mathbb{R}^n$ that gets mapped to $\bar{b}$.

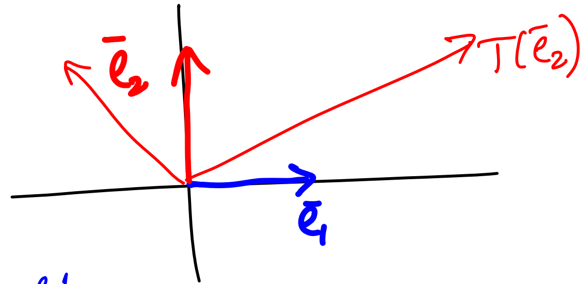
(e) False. Horizontal shear.

$$T(\bar{e}_1) = \bar{e}_1$$

$$T(\bar{e}_2) = \bar{e}_2 + c\,\bar{e}_1 = \begin{bmatrix} c \\ 1 \end{bmatrix}$$

if $c > 0$, we shear to the right

$c < 0$, we shear to the left




$$A = \left[ T(\bar{e}_1)\ T(\bar{e}_2) \right] = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}.$$

$c \in \mathbb{R}$. ($c = 0$ creates no change at all).