MATH 220 – Lecture 11 (09/24/2013)

Midterm on Thursday, Oct 3, during lecture, in TODD 125.

Practice midterm and study guide are posted on the course web page.

The matrix of an LT

Theorem 10 (in the book) $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Then

$T(\bar{x}) = A\bar{x}$, where $A \in \mathbb{R}^{m\times n}$ is

$$A = [T(\bar{e}_1)\ T(\bar{e}_2) \dots T(\bar{e}_n)],$$ where

$$\bar{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \xrightarrow{} j^{th} \text{ position}$$ is the j^{th} unit vector.

Proof idea: Any vector $\bar{x} \in \mathbb{R}^n$ can be written as a unique linear combination of the unit vectors $\bar{e}_j$, $j=1,\dots,n$.

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \text{ or } x_1\bar{e}_1 + x_2\bar{e}_2 + \dots + x_n\bar{e}_n.$$

For an LT T, we have $T(c\bar{u}+d\bar{v}) = cT(\bar{u})+dT(\bar{v})$.

Extending to n vectors (in place of 2), we get that

$$T(c_1\bar{u}_1+c_2\bar{u}_2+\cdots+c_n\bar{u}_n)= c_1T(\bar{u}_1)+c_2T(\bar{u}_2)+\cdots+c_nT(\bar{u}_n).$$ Hence,

$$T(x_1\bar{e}_1+x_2\bar{e}_2+\cdots+x_n\bar{e}_n) = x_1T(\bar{e}_1)+x_2T(\bar{e}_2)+\cdots+x_nT(\bar{e}_n).$$

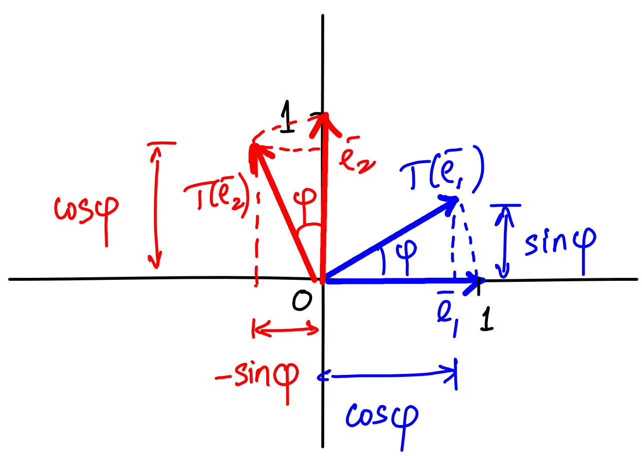
$$= \underbrace{\begin{bmatrix} T(\bar{e}_1) & T(\bar{e}_2) & \cdots & T(\bar{e}_n)\end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{bmatrix}}_{\bar{x}}$$

We illustrate this result by specifying the matrix of several LTs in 2D that have geometric descriptions

Geometric Linear Transformations in 2D

① Rotation by an angle φ (counter clockwise)

$$\bar{e}_1=\begin{bmatrix}1\\0\end{bmatrix}, \quad \bar{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$$



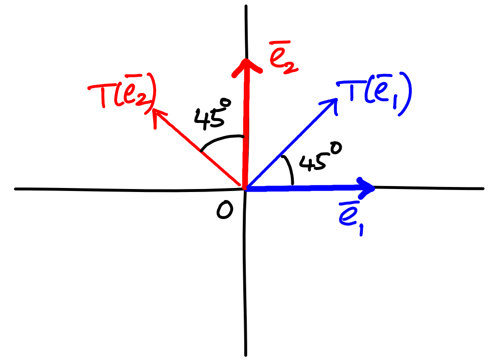
$$T(\bar{e}_1) = \begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix}$$

$$T(\bar{e}_2) = \begin{bmatrix}-\sin\varphi\\ \cos\varphi\end{bmatrix}$$

$$A = \begin{bmatrix}T(\bar{e}_1) & T(\bar{e}_2)\end{bmatrix} = \begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}.$$

e.g., $\varphi = 45^\circ$, $\cos\varphi = \sin\varphi = \frac{1}{\sqrt{2}}$.

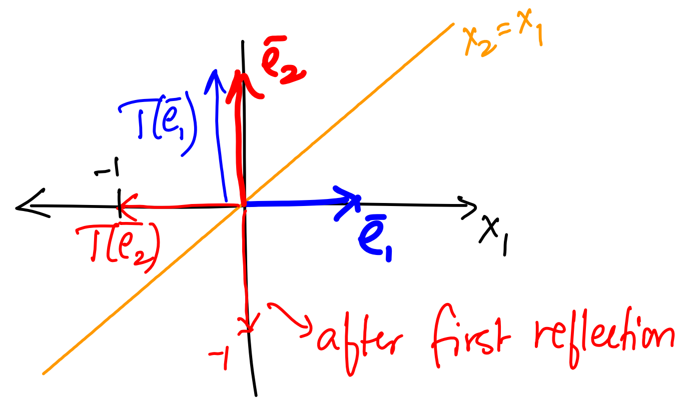
So $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$.



Check out the "Mangle the Coug" link posted on the course web page:
http://www.math.wsu.edu/faculty/hudelson/transform.html

② Reflections

Prob 10, pg 78



$T(\bar{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(\bar{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

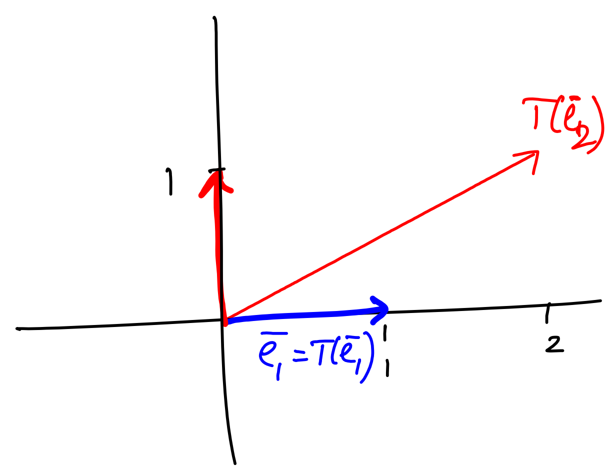
$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

③ Shears

Prob 4, pg 78

$T(\bar{e}_1) = \bar{e}_1$, $T(\bar{e}_2) = \bar{e}_2 + 2\bar{e}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$



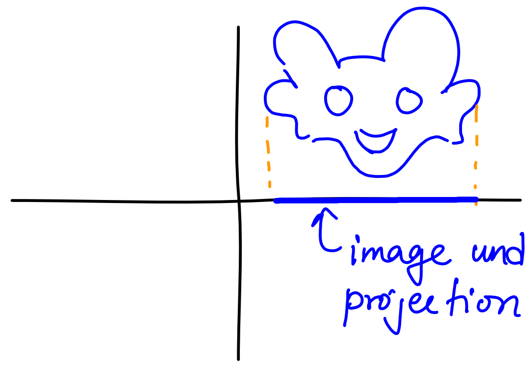
④ Projections

e.g., project vectors onto the horizontal axis.

$$T(\bar{e}_1) = \bar{e}_1 \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T(\bar{e}_2) = \bar{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad T(\bar{x}) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$



The results we have already seen on existence and uniqueness of solutions to $A\bar{x} = \bar{b}$ could be used to answer similar questions in the context of linear transformations. We first define certain types of LTs corresponding to these concepts.

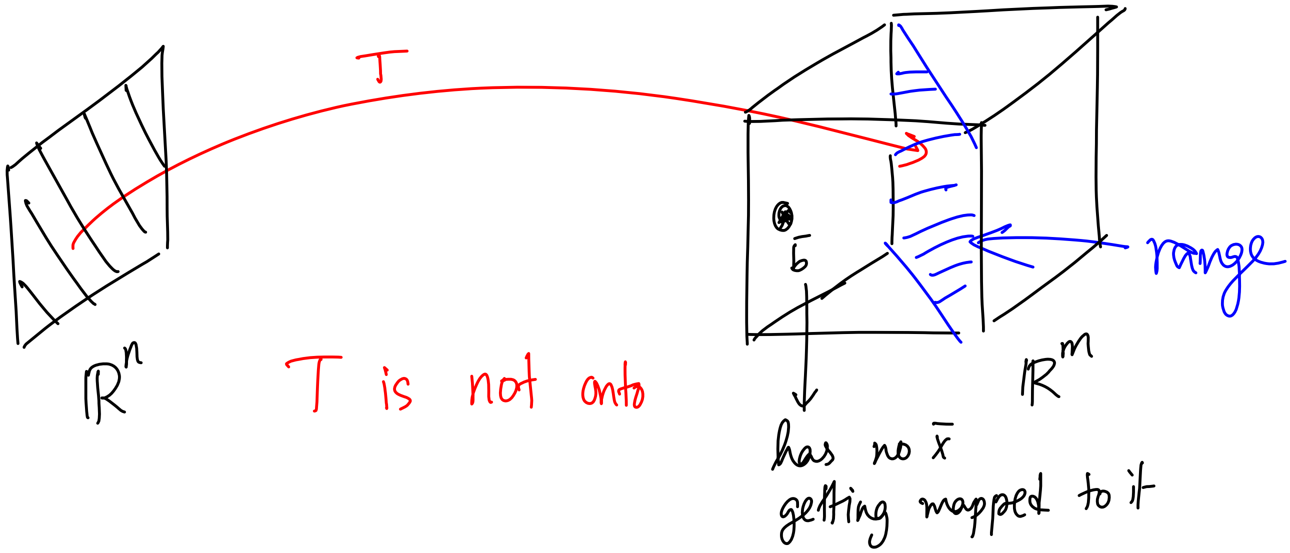
Existence and uniqueness questions for LTs

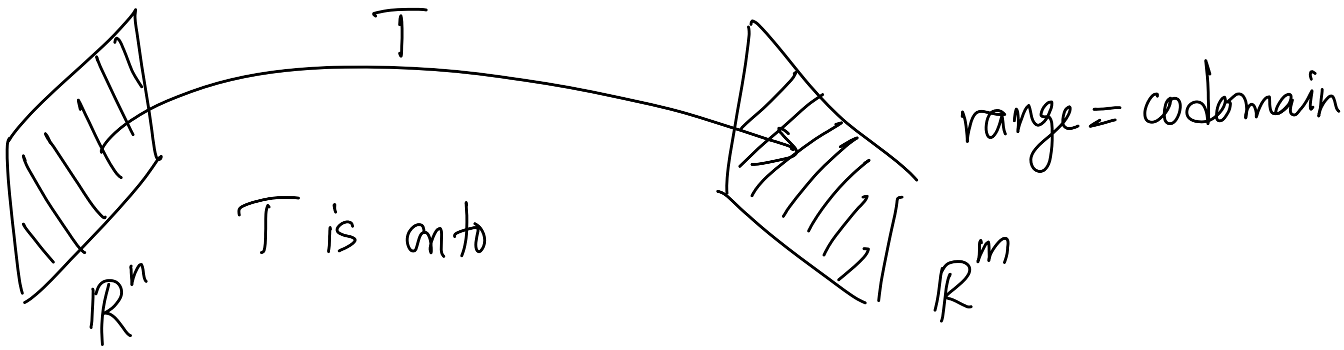
Onto and one-to-one transformations

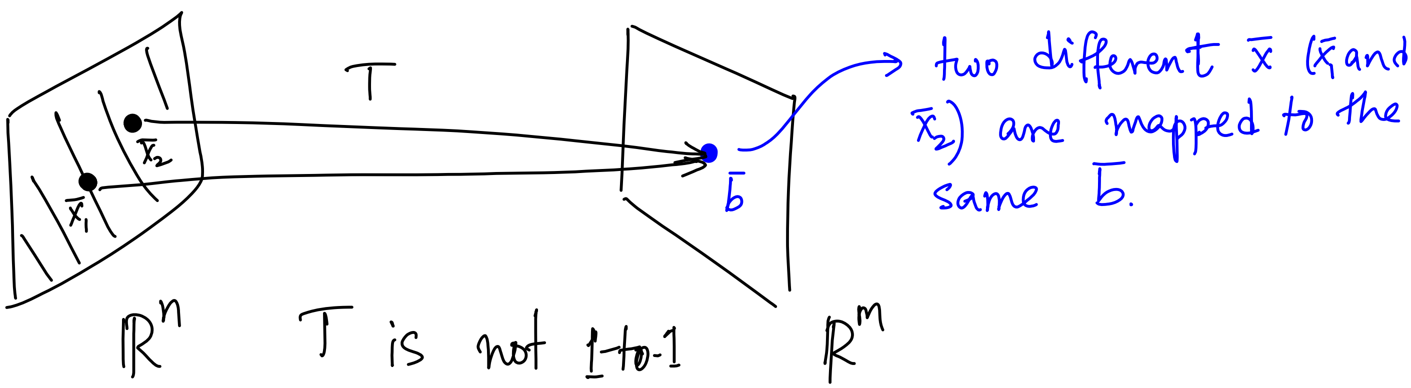
Def $T: \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if each $\bar{b}$ in $\mathbb{R}^m$ is the image of **at least one** $\bar{x}$ in $\mathbb{R}^n$.

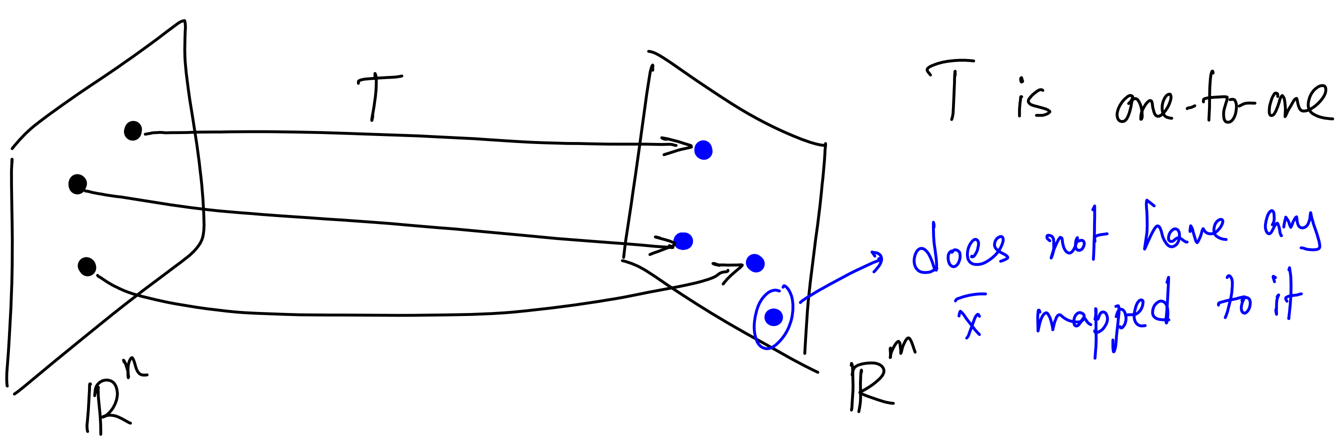
$T: \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if each $\bar{b}$ in $\mathbb{R}^m$ is the image of **at most one** $\bar{x}$ in $\mathbb{R}^n$.

→ some $\bar{b}$ could have no $\bar{x}$ getting mapped to it, in this case.


T
$\bar{b}$
range
$\mathbb{R}^n$
T is not onto
$\mathbb{R}^m$
has no $\bar{x}$ getting mapped to it


T
range = codomain
T is onto
$\mathbb{R}^n$
$\mathbb{R}^m$


T
$\bar{x}_2$
$\bar{x}_1$
$\bar{b}$
two different $\bar{x}$ ($\bar{x}_1$ and $\bar{x}_2$) are mapped to the same $\bar{b}$.
$\mathbb{R}^n$
T is not 1-to-1
$\mathbb{R}^m$


T
T is one-to-one
does not have any $\bar{x}$ mapped to it
$\mathbb{R}^n$
$\mathbb{R}^m$

$T: \mathbb{R}^n \to \mathbb{R}^m \quad T(\bar{x}) = A\bar{x}$

is onto if A has a pivot in every row.

is one-to-one if A has a pivot in every column.

Recall that if A has a pivot in every row, $A\bar{x} = \bar{b}$ is consistent for every $\bar{b} \in \mathbb{R}^m$. Similarly, if A has a pivot in every column, then there cannot exist any free variables, and hence $A\bar{x} = \bar{b}$ has a unique solution, or is inconsistent.