

# SECTION 1.5 : SUMMATION NOTATION + WORK WITH SEQUENCES

Read Section 1.5 (pages 65 – 69)

## Overview

In Section 1.5 we learn to work with summation notation and formulas. We will also introduce a brief overview of sequences, material not included in the text.

## Concept Development

1. In Section 1.5, we introduce the concept of **summation**. I would like to begin with a story about Carl Freidrich Gauss (1777 – 1855), sometimes rated as one of the greatest mathematicians who ever lived. The story goes something like this:

As a youth, Gauss of course attended class with only boys under the tutelage of a headmaster. One day the boys were being particularly unruly, so the master assigned them a task to keep them occupied: “Add the numbers 1 – 100 (that is  $1 + 2 + 3 + \dots + 100$ ).” The boys bent to their task, all but Carl Freidrich Gauss, who put his hand in the air and said “Sir! The answer is 5050.” - - and he was right!

How did he do it? One way is to simply string the numbers out as seen below:

$$1 + 2 + 3 + \dots + 98 + 99 + 100$$

If you look at the first and last you see a sum of 101. If you look at the 2<sup>nd</sup> and the next to last you see  $2 + 99 = 101$ . If you look at the next pairing,  $3 + 98$ , you see a sum of 101! There are 50 pairs in this overall sum and each has a total of 101. The result:  $50 \times 101 = 5050!$

Why is this process valid? If you start at the lower end and move in one entry in the sum, that is, move from 1 to 2, you add 1. If you start at the upper end and move back one in the sum, that is, move from 100 to 99, you lose 1. The loss on the upper end compensates for the increase on the lower end. This process repeats itself as you move from one pairing to another throughout the overall sum. Result: 50 pairs of 101 and total of 5050.

This process works best when you have an even number of terms in your sum.

An alternative approach is to write the sum down twice, once “front to back” and the second time “back to front” as seen below:

$$\begin{array}{r} 1 + 2 + 3 + \dots + 98 + 99 + 100 \\ 100 + 99 + 98 + \dots + 3 + 2 + 1 \end{array}$$

This time we pair the numbers vertically, again obtaining sums of 101. How many such sums do we have? (Answer: 100) Thus, in all the individual sums we have a total of  $100(101) = 10,100$ . However, we have counted each number twice, so, to compensate, we divide by 2. Result? 5050!

This process can be generalized with a sum of the form  $1 + 2 + 3 + \dots + (n - 1) + (n - 2) + n$

where you have  $n$  terms. When you write the sum down twice, once “front to back” and the second time “back to front”, then add vertically, you have  $n$  pairs of  $(n + 1)$  as seen below.

1 +	2 +	3 +	..... +	$(n - 2) +$	$(n - 1) +$	$n$
$n +$	$(n + 1) +$	$(n + 2) +$	..... +	3 +	2 +	1

Total?  $n(n + 1)$ . However we counted each term in our initial sum twice, so, to compensate, we divide by 2, leading to the formula:

$$1 + 2 + 3 + \dots + (n - 1) + (n - 2) + n = \frac{n(n + 1)}{2}$$

Thus, any time we see a sum of this form, we may use the formula, the first of those presented on page 68.

2. We often see sums represented with summation notation. For example the sum of Gauss’ story may be represented as:

$$1 + 2 + 3 + \dots + 98 + 99 + 100 = \sum_{i=1}^{100} i$$

where the symbol  $\sum$  stands for “summation”,  $i$  is known as an **index**, and the numbers 1 and 100 are the **bounds of summation**.

You try one! What sum is represented by  $\sum_{i=2}^{10} j$  ? (Answer:  $2 + 3 + 4 + \dots + 9 + 10$ )

See Examples 1 and 2, pages 66 and 67 for further work with these concepts. Note also properties that apply illustrated on page 67 numbered to the right of the page **6, 7 and 11**.

3. Additional formulas, presented without proof at the top of page 68, are as follows:

$$\sum_{i=1}^n i^2 = 1^2 + 2^2 + \dots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \quad \text{and}$$

$$\sum_{i=1}^n i^3 = 1^3 + 2^3 + \dots + n^3 = \frac{n^2(n + 1)^2}{4}$$

Please see additional work with these formulas and the summation properties in Example 3, pages 68 and 69.

4. Next, we include some additional material not presented in the text. What if we looked at the terms of a summation, but did not add? Such a listing of terms is known as a **sequence** or a **progression**. Examples below:

1, 2, 3, 4, .....      *an infinite sequence since the dots indicate it continues forever*

or

1, 2, 3, 4, . . . . . 99, 100

a sequence representing the terms of the sum we discussed above

Obviously, terms of a sequence may be added. We may use summation notation or formulas and properties discussed above when appropriate. Such a summation of the terms of a sequence is known as a series.

- 5. We are often given a “general term” or “*n*th term” for a sequence. For example, if I told you that you to list entries in a sequence with general term “ $2n$ ”, you would respond with the listing 2, 4, 6, 8 . . . . . and if I asked for entries in a sequence with general term “ $n^2$ ”, you would respond with 1, 4, 9, 16, 25, . . . . . What about the sum of the first 10 entries of such a sequence? Use the formula presented in step 3 above:

$$\frac{10(10 + 1)(20 + 1)}{6} = \frac{10 \times 11 \times 21}{6}$$

You would then evaluate using the arithmetic tools of cancellation and calculation to obtain 385.

- 6. There are two special types of sequences, **arithmetic** and **geometric**. (Note the new use of familiar terminology)

In an arithmetic sequence, we start with an **initial entry**, then **add a common difference** repeatedly obtaining additional entries in the sequence.

- If our initial entry is 2 and our common difference is 3, we would obtain the sequence 2, 5, 8, 11, 14, . . . . .
- If our initial entry was 3 and our common difference is 2, we would obtain the sequence 3, 5, 7, 9, 11, . . . . .

In a geometric sequence, we start with an initial entry, then multiply by a common ratio repeatedly obtaining additional entries in the sequence.

- If our initial entry is 2 and our common ratio is 3, we would obtain the sequence 2, 6, 18, 54, 162, . . . . .
- If our initial entry was 3 and our common ratio is 2, we would obtain the sequence 3, 6, 12, 24, 48, . . . . .

- 7. We are often given an arithmetic or geometric sequence and asked to find the “general” or “*n*th “ term. How do we go about this task?? In a very organized way!

Let’s start with the sequence 1, 3, 5, 7, 9, 11 . . . . . We have shown the first 6 terms in an infinite sequence. The first term is 1, the 2<sup>nd</sup> is 3, the 3<sup>rd</sup> is 5 and so on. Thus, you see that we may number our terms. Let’s put each term vertically into a table, number it and analyze it.

<i>Number of Entry (n)</i>	<i>Term</i>	<i>Analysis</i>	<i>Simplification</i>
1	1	1 + 0	1 + 0(2)
2	3	1 + 2	1 + 1(2)
3	5	1 + 2 + 2	1 + 2(2)
4	7	1 + 2 + 2 + 2	1 + 3(2)
5	9	1 + 2 + 2 + 2 + 2	1 + 4(2)

How can we develop the general term from this information? We note, in the right hand column, that the initial entry (1) and the common difference (2) are constant. The only piece that changes is the coefficient (multiplier) of 2. Can we tie it to anything? Yes! In each case the coefficient is one less than the number of the term! It would appear that our terms are of the form  $1 + (n - 1)2$ . Simplified, our general term is  $1 + 2n - 2$  or  $2n - 1$ .

Lets check this! Our 6<sup>th</sup> term above is 11 - - - and  $2(6) - 1 = 11!$  It appears to work!

Can we carry this forward? What would the 100<sup>th</sup> term in our sequence be?  $2(100) - 1 = 199$ . What would the 1000<sup>th</sup> entry be?  $2(1000) - 1 = 1999!$

7. What about the **general term** of a **geometric sequence**? Let's start with the sequence 3, 6, 12, 24, 48, 96 . . . . . We have shown the first 6 terms in an infinite sequence. The first term is 3, the 2<sup>nd</sup> is 6, the 3<sup>rd</sup> is 12 and so on. Thus, you see that we may number our terms. Let's put each term vertically into a table, number it and analyze it.

<i>Number of Entry (n)</i>	<i>Term</i>	<i>Analysis</i>	<i>Simplification</i>
1	3	3	$3(2^0)$
2	6	$3 \times 2$	$3(2^1)$
3	12	$3 \times 2 \times 2$	$3(2^2)$
4	24	$3 \times 2 \times 2 \times 2$	$3(2^3)$
5	48	$3 \times 2 \times 2 \times 2 \times 2$	$3(2^4)$

How can we develop the general term from this information? We note, in the right hand column, that the initial entry (3) and the common ratio (2) are constant. The only piece that changes is the exponent applied to 2. Can we tie it to anything? Yes! In each case the exponent is one less than the number of the term! It would appear that our terms are of the form  $3(2^{n-1})$ . Lets check this! Our 6<sup>th</sup> term above is 96 - - - and  $3(2^5) = 3 \times 32 = 96!$  It appears to work!

Can we carry this forward? What would the 100<sup>th</sup> term in our sequence be?.  $3(2^{99}) = ? ! ?$  (Let's leave it in exponential form!) What would the 1000<sup>th</sup> entry be?  $3(2^{999})$

### Do you need additional help with these concepts?

Help options:

- Try the practice problems below. Solutions are presented following the assignment.
  - Use a formula presented in Section 1.5 to find  $1 + 2 + 3 + \dots + 45$
  - Evaluate  $\sum_{j=1}^{25} 2j^2$
  - Find the 4<sup>th</sup> term in the sequence with general term  $3(5^{(n-1)})$
  - Find the general term of the arithmetic sequence 4, 7, 10, 13, 15 . . . . .

## Summary

In this section and the additional material, we learned work with summation concepts, formulae and notation and with sequences, using a general term or format to develop a sequence and finding the general term of a given **arithmetic** or **geometric** sequence.

### Assignment:

Complete the following Section 1.5 problems (page 69):

1–15 (odd problems only), 19, plus additional problems below.

1. If a sequence is represented as  $a_i = 2, 3, 4, 5, 6$ , find  $a_3$ .
2. Write the first 5 terms of an arithmetic sequence with first term of 3 and common difference 2.
3. Write the first 5 terms of a geometric sequence with first term 3 and common ratio 2.
4. What are the next 3 entries in the infinite sequence  $5, 10, 17, 26, 37, \dots$ ?
5. a.) Find the general term of the infinite sequence  $3, 8, 13, 18, 23, \dots$ .  
b.) Find the 100<sup>th</sup> term of this sequence.
6. a.) Find the general term of the infinite sequence  $3, 6, 12, 24, 48, \dots$ .  
b.) Find the 100<sup>th</sup> term of this sequence. (*You may leave your answer in exponential form.*)

### Solution to Practice Problem A:

The formula of choice is  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ . In this application, it becomes

$$\sum_{i=1}^{45} i = \frac{45(46)}{2} = 1035.$$

### Solution to Practice Problem B:

The formula of choice is  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ . In this application, we see

$$2 \sum_{i=1}^{25} i^2 = 2 \frac{25(26)(51)}{6} = \frac{25(26)(17)}{1} = 11050. \text{ (Note the use of the basic math technique "cancellation.")}$$

**Solution to Practice Problem C:**

The 4<sup>th</sup> term in the sequence with general term  $3 \times 5^{(n-1)}$  is  $3 \times 5^{(4-1)} = 3 \times 5^3 = 3 \times 125 = 375$

**Solution to Practice Problem D:**

The general term of the arithmetic sequence 4, 7, 10, 13, 15 . . . . . may be found as shown below. First number the terms of the sequence with 4 as the 1<sup>st</sup>, 7 as the 2<sup>nd</sup> and so on, where  $n$  represents the number of the general term.

<i>"n"</i>	<i>Term</i>	<i>Analysis of Term</i>	<i>Simplification</i>
1	4	4 + 0	4 + 0(3)
2	7	4 + 3	4 + 1(3)
3	10	4 + 3 + 3	4 + 2(3)
4	13	4 + 3 + 3 + 3	4 + 3(3)
....	....	.....	.....

Thus we see that each term in the sequence is  $4 + (n - 1)(3)$  where  $n$  is the number of the term. Thus the general term is  $4 + (n - 1)(3) = 4 + 3n - 3 = 3n + 1$ . Check your work. Is  $7 = 3(2) + 1$ ? Yes! Is  $10 = 3(3) + 1$ ? Yes! and so on.