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The well-posedness of a model of an apparatus swimming in the 2-D Stokes fluid.

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Abstract

We discuss a mathematical model of a mechanical device (an "apparatus", a "robotic fish") which can "swim" in the 2-D nonstationary Stokes fluid. We assume that it consists of finitely many subsequently connected "thick (non-zero-measure) points" and that its structure is preserved by the elastic forces acting according to Hooke's Law. Each point forming the apparatus can act upon any of the adjacent points in a "rotation fashion" with the purpose to generate its "fish-like swimming motion". Our goal is to construct this model and to prove the existence and uniqueness result for the resulting nonlinear system of coupled pde's and integro-diferential equations. Models like this are of interest in biology, and also in engineering applications dealing with propulsion systems in fluids.

Key words: Stokes equation, swimming model, coupled system, existence, uniqueness

AMS(MOS) subject classifications. 76, 92, 35.

1. Model description. In this paper we intend to setup and investigate the well-posednes of a simplified model of a mechanical device (a "robotic fish") *swimming-by-itself* (as *opposed to* bodies that are *drifting, or being pushed/pulled by external forces*) in a fluid.

To our surprise we were not able to find a model of this type with investigated well-posedness in the literature, while, on the other hand, various mathematical models of swimming organisms are readily available in the area of mathematical biology, where the research focuses primarily on the computational aspects of the problem, see, e.g., [8], [9], [2], [3], [1].

More precisely, we introduce the following mathematical model, consisting of *two coupled systems of equations* for the *fluid* and for the *position of the apparatus* in it:

$$\frac{\partial y}{\partial t} = \nu\Delta y + F(y,z,v) - \nabla p \quad \text{in} \quad Q_T = \Omega \times (0,T), \tag{1.1}$$

$$\operatorname{div} y = 0 \ \text{ in } \ Q_T, \quad y = 0 \quad \text{in } \Sigma_T = \partial\Omega \times (0,T), \quad y\,|_{t=0} = y_0,$$

$$\frac{dz_i}{dt} = \frac{1}{\text{mes}\,\{S_r(0)\}} \int\limits_{S_r(z_i(t))} y(x,t)dx, \quad z_i(0) = z_{i,0}, \quad i = 1,\ldots,n, \tag{1.2}$$

where

$$z(t) = (z_1(t),\ldots,z_n(t)), \quad v(t) = (v_1(t),\ldots,v_{n-1}(t)),$$

$$\begin{aligned} F(y,z,v) &= \sum_{i=2}^{n+1} \xi_{i-1}(x,t)[k_{i-1}\frac{(\| z_i(t) - z_{i-1}(t) \|_{R^2} - l_{i-1})}{\| z_i(t) - z_{i-1}(t) \|_{R^2}}(z_i(t) - z_{i-1}(t)) \\ &+k_{i-2}\frac{(\| z_{i-2}(t) - z_{i-1}(t) \|_{R^2} - l_{i-2})}{\| z_{i-2}(t) - z_{i-1}(t) \|_{R^2}}(z_{i-2}(t) - z_{i-1}(t))] \\ &+\sum_{i=2}^{n+1} \xi_{i-1}(x,t)\left(v_{i-1}(t)A\left(z_i(t) - z_{i-1}(t)\right) + v_{i-2}(t)A\left(z_{i-2}(t) - z_{i-1}(t)\right)\right). \end{aligned} \tag{1.3}$$

In the above, Ω is a bounded domain in R^2 with boundary $\partial\Omega$ of class C^2, $y = (y_1(x,t), y_2(x,t))$ and $p(x,t)$ denote respectively the velocity and pressure of the fluid at point $x = (x_1,x_2) \in \Omega$ at time t, and ν is a viscosity constant.

At any given moment of time the apparatus looks like a "broken line" structure (or "snake-like"), formed by an ordered sequence of "thick points" $z_1(t)\xi_1(x,t), \ldots, z_n(t)\xi_n(x,t)$, where $z_i(t), i = 1,\ldots,n$ are points in Ω and ξ_i's are the characteristic functions of their supports $\{S_r(z_i(t))\}_{i=1}^n$ of given "small" size r in Ω,

$$\xi_i(x,t) = \begin{cases} 1, & \text{if } x \in S_r(z_i(t)), \\ 0, & \text{if } x \in \Omega\backslash S_r(z_i(t)), \end{cases} \qquad i = 1,\ldots,n. \tag{1.4}$$

We assume that $S_r(0)$ is a given open set, positioned symmetrically in a certain way about the origin within its "small" neighborhood of radius r (accordingly, $S_r(a)$ denotes the same set shifted to point a).

To simplify Σ-notations throughout the paper, we also introduce two auxiliary points z_0 and z_{n+1} as $z_0(t) = z_1(t), \quad z_n(t) = z_{n+1}(t)$.

The term $F(y, z, v)$ in (1.1), (1.3) represents the *internal forces*, generated by the apparatus, acting, in turn, as *external forces* upon the fluid (see Remark 1.2 below for more details). Namely, each of the above mentioned points $z_i(t)$ can force any of the adjacent points to rotate about it, as described in the last line on the right in (1.3), where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The magnitudes of the applied rotation forces are determined by the coefficients $v_i, i = 1, \ldots, n-1$, which can be viewed as *bilinear* controls (see, e.g., [4]-[6]).

The structure of the apparatus as a whole is preserved by the elastic forces which act according to Hooke's Law when the distances between adjacent points deviate from the preassigned values

$$l_{i-1} = \| z_i(0) - z_{i-1}(0) \|_{R^2} > 0, \quad i = 2, \ldots, n, \tag{1.5}$$

as described in the first two lines on the right in (1.3), where the given parameters $k_i > 0$ describe the rigidity of the link $z_{i-1}(t)z_i(t)$, $i = 2, \ldots, n$. (For the auxiliary points we set $k_0 = k_n = l_0 = l_n = 0$.)

Remark 1.1.

- Note that, when the (actual) adjacent points in the apparatus share the same position in space, the forcing term F in (1.3) "blows up", and hence the model (1.1)-(1.3) become mathematically undefined. This issue is of principal importance in our further discussion of the well-posedness of this model.

- Assuming that r in (1.4) is "small", in the above *we modeled all the apparatus' forces in (1.3) as of the same value within their respective ("small") supports* $S_r(z_i(t)), i = 1, \ldots, n$.

In our model the ("small") thick points $z_i(t)\xi_i(x,t), i = 1, \ldots, n$ are treated as massless or immaterial. Their dynamics are determined by the resulting average motion of the fluid within the respective supports $S_r(z_i(t)), i = 1, \ldots, n$, as described in (1.2).

The model (1.1)-(1.3) can be viewed as a "hybrid" model in the sense that we model an apparatus as a collection of finitely many points with flexible immaterial internal links (or, say, which have a "negligable affect" on the fluid), each of which is surrounded by "very small" massless support. In other words, we consider our apparatus as a "part of fluid" - as opposed to a mechanism composed out of solid bodies. Such approach is often used in various "swimming" biological models (see, e.g., [8], [9], [2], [3], [1]) and, we believe, it can be useful to apprach the issue of controllability of the swimming process as well.

Remark 1.2: Internal forces and conservation of momentums.

- We want to emphasize that all forces in (1.3) satisfy the 3-rd Newton's Law and their sum is equal to zero. Thus, they are *internal with respect to the apparatus and cannot move its center of mass.* This is the principal property of our model as a "swimming-by-itself-device".

- In our model only the adjacent points in the apparatus are allowed to interact. Since both (internal) rotation and elastic forces between any such points satisfy Newton's 3-rd law, the linear momentums generated by such forces are conserved (see, e.g., [10]). However, the rotation forces, say, between the thick points $z_i(t)\xi_i(x,t)$ and $z_{i+1}(t)\xi_{i+1}(x,t)$ generate a non-zero torque. This means that, to have the conservation of the angular momentums, we need to assume that there are some other, also internal forces in the apparatus (from an "engine") that generate the corresponding "negating" torque. (Mathematically, such forces are to be described by a separated system of equations, *decoupled* from the model (1.1)-(1.3), whose solutions provide the values for $v_1, \ldots, v_{n-1}$.)

  For example, we can view any pair of two thick points $z_i(t)\xi_i(x,t)$ and $z_{i+1}(t)\xi_{i+1}(x,t)$ as two "floating" platforms with shafts attached at their respective centers of mass in such a way that these shafts can freely rotate about their axes without causing the rotations of the platforms themselves. Note, however, that a force applied to such shaft (i.e., to the respective center of mass) would make the platform at hand carring it move in the fluid (with no rotation). Suppose that these shafts are connected by a solid link (say, above the surface of the water, so it does not interact with the fluid). If we further assume that there is a couple of anti-symmetric forces that being applied to the shaft on the platform $z_i(t)\xi_i(x,t)$ make it rotate, then the link between the shafts will generate the internal forces between the platforms $z_i(t)\xi_i(x,t)$ and $z_{i+1}(t)\xi_{i+1}(x,t)$ as described in (1.3).

The aforementioned couple of anti-symmetric forces can be generated, e.g., by a couple of springs, attached in an "anti-symmetric" fashion to the shaft at the center of mass of $z_i(t)\xi_i(x,t)$ at one ends and to the opposite sides of a disc ("ring") which can freely rotate about the shaft as well (similar to the watch-and-a-watch's hand mechanism). (Alternatively, one can simply imagine something like two rocket engines positioned anti-symmetrically about the shaft).

2. Existence and uniqueness. Let $\dot{J}(\Omega)$ denote the linear space of infinitely differentiable 2-D vector functions $\phi(x) \in R^2$ which have compact support in Ω and are solenoidal (or divergence free), that is, $\operatorname{div}\phi = 0$ in Ω. By $H(\Omega)$ we denote the completion of this space in the norm

$$\| \phi \|_{H(\Omega)} = \left(\int_\Omega (\| \phi_{x_1} \|_{R^2}^2 + \| \phi_{x_2} \|_{R^2}^2) dx \right)^{1/2}.$$

We decompose (e.g., [7, 11]) the vector space $(L^2(Q_T))^2$ into two orthogonal subspaces $J_0(Q_T)$ and $G(Q_T)$ assuming that the elements of the former belong to the completion $J_0(\Omega)$ of $\dot{J}(\Omega)$ in the norm of $(L^2(\Omega))^2$ and the elements of the latter to its orthogonal complement $G(\Omega)$ in this space for almost all $t \in (0,T)$.

Assumption 2.1. *Assume that*

$$l_{i-1} > 0, \quad i = 2, \dots, n; \quad \bar{S}_r(z_i(0) \subset \Omega, \quad i = 1, \dots, n; \tag{2.1}$$

and the set $S_r(0)$ is such that

$$\int_{(S_r(0) \bigcup S_r(h)) \setminus (S_r(0) \bigcap S_r(h))} dx = \int_\Omega | \xi(x) - \xi(x-h) | \, dx \leq Ch \quad \forall h \in (-h_0, h_0) \tag{2.2}$$

for some positive constants h_0 and C, where $\xi(x)$ is the characteristic function of $S_r(0)$.

Conditions (2.1) simply mean that our apparatus lies in Ω at time $t = 0$ and that the initial positions of any two adjacent points forming it are distinct (see also Remark 2.1 for alternative assumptions). In general, in typical applications, we need them all distinct but this is not essential for the mathematical issues we address here. Condition (2.2) is not difficult to satisfy - it holds, e.g., for rectangles and circles.

Principal difficulty: Blow-up and controllability. For the issue of well-posedness of system (1.1)-(1.3), the fact that the *forcing term* F *"blows up,"* when any of the adjacent points in the apparatus share the same position, poses a principal difficulty. Indeed, clearly, depending on what initial datum and values for v_is are given in (1.1)-(1.3), such situation seems ("physically") plausible at some future moment (or for some postitive duration of time), even if solution to this system exists on some "small" time-interval. On the other hand, it does not have necessarily to happen. The former issue, namely, the existence on some "small" $(0, T)$, is addressed below in our main result Theorem 2.1. The latter issue can be viewed as the issue of *"global" controllability* (which is of our further interest), namely, when one tries to select multiplicative controls v_i's with the purpose to ensure that the apparatus is *"swimming" in desirable fashion, while avoiding the aforemention "blow-up."*

We have the following result.

Theorem 2.1. *Let* $y_0 \in H(\Omega)$; $T^* > 0; k_i > 0$, $v_i \in L^\infty(0, T^*), i = 1, \ldots, n-1$*; and* $z_{i,0} \in \Omega, i = 1, \ldots, n$ *be given and Assumption 2.1 hold. Then there exists a* $T = T(z_{1,0}, \ldots, z_{n,0}, \| v_1 \|_{L^\infty(0,T^*)}, \ldots, \| v_{n-1} \|_{L^\infty(0,T^*)}, \Omega) \in (0, T^*)$ *such that system (1.1)-(1.5), (2.1), (2.2) admits a unique solution* $\{y, p, z\}$ *on* $(0, T)$, $\{y, \nabla p, z\} \in J_0(Q_T) \times G(Q_T) \times [C([0, T]; R^2)]^n$*. Moreover,* $y \in C([0, T]; H(\Omega))$, $y_t, y_{x_i x_j} \in (L^2(Q_T))^2$, $p_{x_i} \in L^2(Q_T)$, *where* $i, j = 1, 2$, *and equations in (1.1) and (1.2) are satisfied almost everywhere, while*

$$z_i(t) \neq z_{i+1}(t), \;\; i = 1, \ldots, n-1; \quad \bar{S}_r(z_i(t) \subset \Omega, \quad t \in [0, T], \;\; i = 1, \ldots, n. \qquad (2.3)$$

Remark 2.1.

- Conditions (2.3) mean that at any moment for the solution of (1.1)-(1.5), (2.1), (2.2), whose existence is established in Theorem 1.1, we can guarantee (by choosing "small enough" time T) that any two adjacent points in the apparatus do not share the same point in space, while our swimming device stays sufficiently away from its boundary. The former allows us to maintain the wellposedness (both mathematical and "physical") of the elastic forces in (1.3), while the latter implies that we do not have to deal with any "complications" arising when some of the "thick points" "hit" $\partial\Omega$.

- On the other hand, Theorem 2.1 allows sharing of the same space for some portions of supports of the aforementioned thick points (recall they are assumed to be immaterial). Of course, at no extra cost, we could equally make the assumption (2.1) more strict to exclude the latter possibility by assuming that l_i's strictly exceed $2r$ or even to assume that (2.1) (and then (2.3)) holds with margin exceeding $2r$ for all $z_i(t)$'s, i.e., *not only adjacent*, while modifying the statement of Theorem 2.1 accordingly.

As it follows from the proof below, the duration of the time-interval $(0,T)$ in Theorem 2.1 is not quite of local nature. Namely, based on suitable a priori estimates, the value of T is selected small enough to guarantee that conditions (2.3) hold, i.e., that the adjacent points $z_i(t), i = 1,\ldots,n$ remain "sufficiently far away" from each other and from $\partial\Omega$ during the duration of $(0,T)$ for the given choice of data in (1.1)-(1.3). However, depending on what data we have at time T and beyond (namely, $z_i(T), i = 1,\ldots,n$ and v_i's), this solution can be extended further in time as long as (2.3) holds.

3. Proof of Theorem 2.1: Decoupled system (1.2). Our plan to prove Theorem 2.1 is as follows:

- In this section, by the classical methods, we will prove the existence and uniqueness of solution to the decoupled version of system (1.2), namely:

$$\frac{dw_i}{dt} = \frac{1}{\text{mes}\,\{S_r(0)\}} \int\limits_{S_r(w_i(t))} u(x,t)dx, \quad w_i(0) = z_{i,0}, \quad i = 1,\ldots,n, \tag{3.1}$$

  where $u(x,t)$ is some given function from $L^2(0,T;R^2)$.

- In Section 4, based on the properties of solutions of (3.1) and of the Stokes equation, we will apply a fixed point argument to prove that the (1.1)-(1.5), (2.1), (2.2) has a solution with required regularity properties.

- In Section 5 we will show that the found solution is unique, which is possible, among other things, due to the assumption (2.2) on $S_r(0)$.

- The *specific additional issue* with which we have to deal, while implementing this strategy, is that we have to avoid the blow-up of the forcing term F in (1.3). To this end, we will need to derive suitable *a priori estimates* for solutions of decoupled system (3.1) and for the fluid equation (1.1). These estimates should be such that we would be able to carry them over to the original coupled problem (1.1)-(1.3) along the fixed point argument, while ensuring (2.3) to avoid the blow-up of the forcing term (1.3).

Lemma 3.1. *Let $T > 0$ and $u \in L^2(0,T;R^2)$ be given. Then for any set of initial conditions $z_{i,0}, i = 1,\dots,n$ system (3.1) has a solution in $C([0,T];R^2)$. If we assume that (2.1) holds, then $T > 0$ can be selected so that (2.3) holds for this solution as well.*

Proof of Lemma 3.1. We simply apply the classical approach here, namely, we intend to construct a sequence of functions associated with (3.1) and to show that it contains a subsequence converging to a solution of (3.1). While doing this, we will need to obtain suitable estimates to ensure the estimate of type (2.3) for our soultion.

Step 1. Select any $T > 0$. Let $w_i^{(0)}(t) = z_{i,0}$ and

$$w_i^{(1)}(t) = z_{i,0} + \frac{1}{\text{mes}\,\{S_r(0)\}} \int_0^t \int_{S_r(z_{i,0})} u(x,\tau)\,dx\,d\tau, \quad t \in [0,T], \; i = 1,\dots,n.$$

Then we use the induction. Namely, given $w_i^{(k-1)}(t)$, $i = 1,\dots,n$, we construct

$$w_i^{(k)}(t) = z_{i,0} + \frac{1}{\text{mes}\,\{S_r(0)\}} \int_0^t \int_{S_r(w_i^{(k-1)}(\tau))} u(x,\tau)\,dx\,d\tau, \quad t \in [0,T], \; i = 1,\dots,n. \tag{3.2}$$

Note that $w_i^{(k)}$'s also satisfy the decoupled equations:

$$\frac{dw_i^{(k)}}{dt} = \frac{1}{\text{mes}\,\{S_r(0)\}} \int_{S_r(w_i^{(k-1)}(t))} u(x,t)\,dx, \quad w_i^{(k)}(0) = z_{i0}, \quad i = 1,\dots,n. \tag{3.3}$$

Step 2. Next we will show that $\{w_i^{(k)}\}_{k=1}^{\infty}$ is uniformly bounded and equicontinuous in $C([0,T];R^2)$ and, hence, due to Ascoli's Theorem, this sequence contains a converging subsequence in this space.

Indeed,

$$\| w_i^{(k)}(t) \|_{R^2} \leq \| z_{i,0} \|_{R^2} + \frac{1}{\text{mes}\,\{S_r(0)\}} \| \int_0^t \int_{S_r(w_i^{(k-1)}(\tau))} u(x,\tau)dxd\tau \|_{R^2}$$

$$\leq \| z_{i,0} \|_{R^2} + \frac{\sqrt{T}}{\sqrt{\text{mes}\,\{S_r(0)\}}} \| u \|_{L^2(0,T;R^2)} \quad \forall t \in [0,T], \quad k = 1, \ldots; \; i = 1, \ldots, n. \tag{3.4}$$

Selection of T to deal with (2.3). Note that, if (2.1) holds, (3.4) implies that we can select $T > 0$ such that for any $t \in [0,T]$ all $w_i^{(k)}(t)$'s will stay *close enough to their initial values* $z_{i,0}$*'s to satisfy the estimate of type (2.3).*

To show the equicontinuity, notice that

$$\| w_i^{(k)}(t+h) - w_i^{(k)}(t) \|_{R^2} \leq \frac{1}{\text{mes}\,\{S_r(0)\}} \| \int_t^{t+h} \int_{S_r(w_i^{(k-1)}(\tau))} u dxd\tau \|_{R^2}$$

$$\leq \frac{\sqrt{h}}{\sqrt{\text{mes}\,\{S_r(0)\}}} \| u \|_{L^2(0,T;R^2)} \quad \forall t \in [0,T], \quad k = 1, \ldots; \; i = 1, \ldots, n. \tag{3.5}$$

Thus, we showed that $\{w_i^{(k)}\}_{k=1}^{\infty}$ contains a converging subsequence in $C([0,T];R^2)$. Without loss of generality, to simplify notations, we can further assume that

$$w_i^{(k)} \to w_i \text{ in } C([0,T];R^2) \text{ as } k \to \infty, \quad i = 1, \ldots, n. \tag{3.6}$$

Step 3. To show that the limit functions in (3.6) solve (3.1), it is sufficient to pass to the limit in (3.2), noticing that

$$\int_0^t \int_{S_r(w_i^{(k-1)}(\tau))} u dxd\tau = \int_0^t \int_{S_r(w_i(\tau))} u dxd\tau$$

$$+ \int_0^t \int_{S_r(w_i^{(k-1)}(\tau))} u dxd\tau - \int_0^t \int_{S_r(w_i(\tau))} u dxd\tau$$

and then that

$$
\| \int_0^t \int_{S_r(w_i^{(k-1)}(\tau))} u dx d\tau - \int_0^t \int_{S_r(w_i(\tau))} u dx ds\tau \|_{R^2}
$$

$$
\leq \| \int_0^t \int_\Omega u \left(\xi(x, S_r(w_i^{(k-1)}(\tau)) - \xi(x, S_r(w_i(\tau)) \right) dx d\tau \|_{R^2}
$$

$$
\leq \| u \|_{L^2(0,T;R^2)} \left(\int_0^t \int_{R^2} \left(\xi(x) - \xi(x - (w_i^{(k-1)}(\tau) - w_i(\tau))) \right)^2 dx d\tau \right)^{1/2} \tag{3.7}
$$

for all $t \in [0, T], i = 1, \ldots,$ where $\xi(x, S)$ stands for the characteristic function of the set $S \subset R^2$ and $\xi(x) = \xi(x, S_r(0))$ as in (2.2).

Recall now the continuity property of the elements in $L^2(R^2)$, namely, that for any its element, say, g,

$$
\int_{R^2} (g(x) - g(x+h))^2 dx \quad \rightarrow \quad 0 \text{ as } h \rightarrow 0. \tag{3.8}
$$

Since, $w_i^{(k-1)}$'s converge to w_i's in $C([0, T; R^2)$ as $k \rightarrow \infty$, (3.8), applied with $g(x) = \xi(x)$ to (3.7), yields that the last term in (3.7) converges to zero as k tends to ∞. Therefore, (3.2) converge to

$$
w_i(t) = z_{1,0} + \frac{1}{\text{mes}\{S_r(0)\}} \int_0^t \int_{S_r(w_i(\tau))} u dx d\tau, \quad t \in [0, T], \ \ i = 1, \ldots, n, \tag{3.9}
$$

which is equivalent to (3.1). This proves existence of a solution to (3.1) and ends the proof of Lemma 3.1.

Lemma 3.2. *Let* $T > 0$ *and* $u \in (L^2(0, T; L^\infty(R)))^2 \bigcap L^2(0, T; R^2)$ *be given. Then (3.1) admits a unique solution in* $C([0, T]; R^2)$.

Proof of Lemma 3.2. Existence follows from Lemma 3.1. To prove uniqueness of solutions to (3.1), assume the opposite. Let, say, we have two different solutions $w_{11}(t)$ and $w_{12}(t)$ to the first system in (3.1) (for $i = 1$). Without loss of generality, we can assume that they are different right from $t = 0$. Then, similar to (3.7),

$$
\| w_{11}(t) - w_{12}(t) \|_{R^2}
$$

$$\leq \frac{1}{\mathrm{mes}\,\{S_r(0)\}} \parallel u \parallel_{(L^2(0,T;L^\infty(R)))^2} \left(\int_0^t \left[\int_\Omega \mid \xi(x) - \xi(x-(w_{11}(\tau)-w_{12}(\tau)) \mid dx \right]^2 d\tau \right)^{1/2}$$

$$\leq \frac{\sqrt{T_0}}{\mathrm{mes}\,\{S_r(0)\}} \parallel u \parallel_{(L^2(0,T;L^\infty(R))^2} C \parallel w_{11} - w_{12} \parallel_{C([0,T_0];R^2)} \quad \forall t \in [0,T_0] \subseteq [0,T], \quad (3.10)$$

where $T_0 > 0$ is sufficiently small so that we can make use of (2.2) (this is possible, because $w_{11}(0) = w_{12}(0)$). After taking maximum over $[0,T_0]$ in the left-hand side of (3.10), we obtain that

$$\parallel w_{11} - w_{12} \parallel_{C([0,T_0];R^2)} \leq \frac{C\sqrt{T_0}}{\mathrm{mes}\,\{S_r(0)\}} \parallel u \parallel_{(L^2(0,T;L^\infty(R))^2} \parallel w_{11} - w_{12} \parallel_{C([0,T_0];R^2)} . \quad (3.11)$$

This inequality implies that $w_{11} \equiv w_{12}$ on $[0,T_0]$. Contradiction. This ends the proof of Lemma 3.2.

4. Proof of Theorem 2.1: Existence. First we intend to prove existence, making use of the results of the previous section and a fixed point argument.

Step 1. Let $B_q(0)$ denote a ball of radius q (its value will be selected below) with center at the origin in the Banach space $J_0(Q_T) \bigcap L^2(0,T;H^2(\Omega))$ endowed with the norm of $L^2(0,T;H^2(\Omega))$,

$$B_q(0) = \{\phi = (\phi_1,\phi_2) \mid \phi \in J_0(Q_T) \bigcap L^2(0,T;H^2(\Omega)),$$
$$\int_0^T \int_\Omega \left(\parallel \nabla\phi \parallel^2_{R^2} + \sum_{k=1}^2 (\phi_k^2 + \sum_{l,m=1}^2 \phi^2_{kx_lx_m}) \right) dx \leq q^2\}.$$

Select any $T > 0$ "small enough" so that Lemmas 3.1, 3.2 and (2.3) (see Lemma 3.1) hold true for any $u \in B_q(0)$. This is possible due to the upper bound like in (3.4) applied to (3.9) in place of (3.2) uniform with respect to the $L^2(0,T;R^2)$-norm of u and to the fact that $H^2(\Omega)$ is continuously embedded into $C(\bar{\Omega})$ and, hence, $L^2(0,T;H^2(\Omega))$ is continuously embedded into $(L^2(0,T;L^\infty(\Omega)))^2$.

Step 2: Continuity and compactness of solution mapping for (3.1). We now intend to show that the operator

$$A : B_q(0) \ni u \rightarrow \mathbf{w} = (w_1,\ldots,w_n) \in C([0,T];R^2),$$

where w_is are solutions to (3.1), is continuous and compact, if $T > 0$ is sufficiently small.

First we will show that A is continuous. To this end, we will evaluate $\| A(u_1) - A(u_2) \|_{C([0,T];R^2)}$, where $u_1, u_2 \in B_q(0)$.

Let $A(u_j) = (w_1^{(j)}, \ldots, w_n^{(j)}), j = 1, 2$. Then from (3.9), making use of calculations similar to (3.4) and (3.7), (3.10), (3.11), we derive:

$$\| w_i^{(1)}(t) - w_i^{(2)}(t) \|_{R^2} \leq \| \frac{1}{\text{mes}\,\{S_r(0)\}} \int_0^t \int_{S_r(w_i^{(1)}(\tau))} (u_1 - u_2) dx d\tau$$

$$+ \frac{1}{\text{mes}\,\{S_r(0)\}} \int_0^t \int_\Omega u_2(x,\tau) \left(\xi(x, S_r(w_i^{(1)}(\tau)) - \xi(x, S_r(w_i^{(2)}(\tau)) \right) dx d\tau \|_{R^2}$$

$$\leq \frac{\sqrt{T}}{\text{mes}^{1/2}\,\{S_r(0)\}} \| u_1 - u_2 \|_{(L^2(Q_T))^2}$$

$$+ \frac{C\sqrt{T}}{\text{mes}\,\{S_r(0)\}} \| u_2 \|_{(L^2(0,T;L^\infty(\Omega)))^2} \| w_i^{(1)} - w_i^{(2)} \|_{C([0,T];R^2)}, \tag{4.1}$$

where we again used the fact that $L^2(0,T;H^2(\Omega))$ is continuously embedded into $(L^2(0,T;L^\infty(\Omega)))^2$, i.e.,

$$\| \phi \|_{(L^2(0,T;L^\infty(\Omega)))^2} \leq K \| \phi \|_{L^2(0,T;H^2(\Omega))}$$

for some $K > 0$.

Then, for sufficiently small $T = T(q) > 0$ (also satisfying our choice in Step 1),

$$\| w_i^{(1)} - w_i^{(2)} \|_{C([0,T];R^2)} \leq \left(1 - \frac{CKq\sqrt{T}}{\text{mes}\,\{S_r(0)\}} \right)^{-1} \frac{\sqrt{T}}{\text{mes}^{1/2}\,\{S_r(0)\}} \| u_1 - u_2 \|_{(L^2(Q_T))^2} \tag{4.2}$$

and therefore A is continuous for sufficiently small $T > 0$ on $B_q(0)$.

Furthermore, the same argument as in step 2 of the proof of Lemma 3.1 applies to show that the set of all solutions to (3.1) with u lying in any given set bounded in $J_0(Q_T)$ is uniformly bounded and equicontinuos. Hence, A is also compact in $J_0(Q_T)$ and, similar to (3.4),

$$\| w_i \|_{C([0,T];R^2)} \leq \| z_{i,0} \|_{R^2} + \frac{q\sqrt{T}}{\sqrt{\text{mes}\,\{S_r(0)\}}}$$

$$\leq \max_{i=1,\ldots,n} \{\| z_{i,0} \|_{R^2}\} + \frac{q\sqrt{T}}{\sqrt{\text{mes}\,\{S_r(0)\}}} \quad \forall i = 1, \ldots, n, \;\; u \in B_q(0). \tag{4.3}$$

Step 3: Solution mapping for the decoupled Stokes equation. Consider the following decoupled Stoke's initial boundary value problem:

$$\frac{\partial y_*}{\partial t} = \nu \Delta y_* + f(x,t) - \nabla p_* \quad \text{in} \quad Q_T, \tag{4.4}$$

$$\operatorname{div} y_* = 0 \ \text{ in } \ Q_T, \quad y_* = 0 \quad \text{in } \Sigma_T, \quad y_* \mid_{t=0} = y_0 \in H(\Omega).$$

It is known (e.g., [7], Ch. 4) that for any $f \in (L^2(Q_T))^2$ and $y_0 \in H(\Omega)$ (4.4) admits a unique solution in $J_0(Q_T) \bigcap L^2(0,T;H^2(\Omega))$ with the properties described in Theorem 2.1. Moreover (see, e.g., (7) on p. 79 and (49)-(50) on p. 65 in [7], or [11], p. 33),

$$\| y_* \|^2_{L^2(0,T;H^2(\Omega))} \leq L \| y_0 \|^2_{H(\Omega)} + L \int_{Q_T} f^2 dx\, d\tau, \tag{4.5}$$

where L is some positive constant. This means that, given y_0, the operator

$$B : (L^2(Q_T))^2 \ni f \ \to y_* \in L^2(0,T;H^2(\Omega)),$$

is continuous.

Given $u \in B_q(0)$, denote by $F_*(\mathbf{w})$ the value of the term of $F(y,z,v)$ in (1.3) with y_* from (4.4) and w_i's from (3.1) in place of y and z_i's respectively. Then, in view of selection of T in Steps 1 and 2 of this section, similar to (2.3),

$$\| w_{i-1}(t) - w_i(t) \|_{C([0,T];R^2)} > D(T), \quad \forall i = 2,\ldots,n, \ \ u \in B_q(y_0), \tag{4.6}$$

where $D(T) > 0$ does not decrease as $T \to 0+$.

Therefore, the operator

$$F : (C([0,T];R^2))^n \ni \mathbf{w} \ \to \ F_*(\mathbf{w}) \in (L^2(Q_T))^2$$

is continuous for any $u \in B_q(0)$. Moreover, as it follows from (1.2), (4.3), (4.6),

$$\| F_*(\mathbf{w}) \|_{(L^2(Q_T))^2}$$

$$\leq 2n\sqrt{T\mathrm{mes}\,\{\Omega\}} \max_{i=1,\ldots,n}\{k_i\} \left(2 \max_{i=1,\ldots,n} \{\| z_{i,0} \|_{R^2}\} + \frac{2q\sqrt{T}}{\sqrt{\mathrm{mes}\,\{S_r(0)\}}} + \max_{i=1,\ldots,n-1}\{l_i\} \right)$$

$$+4n\sqrt{T\mathrm{mes}\,\{\Omega\}} \max_{i=1,\ldots,n-1}\{\| v_i \|_{L^\infty(0,T)}\} \left(\max_{i=1,\ldots,n}\{\| z_{i,0} \|_{R^2}\} + \frac{q\sqrt{T}}{\sqrt{\mathrm{mes}\,\{S_r(0)\}}} \right). \tag{4.7}$$

Step 4: The fixed point argument. Combining steps 2 and 3, we obtain that the operator

$$BFA : B_q(0) \ni u \to y_* \in J_0(Q_T) \bigcap L^2(0,T;H^2(\Omega))$$

is continuous and compact.

Let us select now the value for q as any positive number exceeding $\sqrt{L} \parallel y_0 \parallel_{H(\Omega)}$ (see (4.5)).

Since all the above holds for $T > 0$ selected in Steps 1 and 2 and for any $T_0 \in (0,T)$ in its place, without loss of generality, in view of estimates (4.5), applied with $f = F_*$, and (4.7), we can assume that our $T > 0$ is so small that the continuous and compact operator BFA maps the closed ball $B_q(0)$ in itself in $L^2(0,T;H^2(\Omega)) \bigcap J_0(Q_T)$. Due to Schauder's fixed point theorem, this operator has a fixed point y, which is a solution to (1.1) and satisfies all the properties in Theorem 2.1. Note that, as usual, ∇p can be selected in the orthogonal complement $G(Q_T)$ of $J_0(Q_T) \in (L^2(Q_T))^2$ to deal with the part of (1.1) that lies in $G(Q_T)$, which will give us p to complement y in Theorem 2.1. In turn, this y also generates the corresponding solution for (1.2). This proves the existence of solution to (1.1)-(1.5), (2.1)-(2.3) on some time interval $(0,T)$.

5. Proof of Theorem 2.1: Uniqueness. Let assume the opposite. Namely, that, e.g., starting at $t = 0$, there are two different solutions $\{z^{(1)} = (z_1^{(1)}, \ldots, z_n^{(1)}), y_1, p^{(1)}\}$ and $\{z^{(2)} = (z_1^{(2)}, \ldots, z_n^{(2)}), y_2, p^{(2)}\}$ to (1.1)-(1.5), (2.1)-(2.3) satisfying the regularity properties described in Theorem 2.1 on some time interval $(0,T)$.

Step 1. Then, as in (4.1)-(4.2), we have:

$$\parallel z_i^{(1)} - z_i^{(2)} \parallel_{C([0,T_0];R^2)} \leq \left(1 - \frac{CKq\sqrt{T_0}}{\text{mes}\,\{S_r(0)\}}\right)^{-1} \frac{\sqrt{T_0} \parallel y_1 - y_2 \parallel_{(L^2(Q_T))^2}}{\text{mes}^{1/2}\,\{S_r(0)\}} \qquad (5.1)$$

for any $T_0 \in (0,T)$ such that

$$\frac{CKq\sqrt{T_0}}{\text{mes}\,\{S_r(0)\}} < 1. \qquad (5.2)$$

Step 2. We will need now a similar estimate for $(y_1 - y_2)$.

Note first that

$$\frac{\partial(y_1 - y_2)}{\partial t} = \nu\Delta(y_1 - y_2) + (F(y_1, z^{(1)}, v) - F(y_2, z^{(2)}, v)) - \nabla(p^{(1)} - p^{(2)}) \quad \text{in } Q_T,$$

$$\operatorname{div}(y_1 - y_2) = 0 \text{ in } Q_T, \quad (y_1 - y_2) = 0 \quad \text{in } \Sigma_T, \quad (y_1 - y_2)\mid_{t=0} = 0,$$

Then, as in (4.5),

$$\| y_1 - y_2 \|^2_{(L^2(Q_T))^2} \leq L \| F(y_1, z^{(1)}, v) - F(y_2, z^{(2)}, v) \|^2_{L^2(Q_{T_0})} .$$

In turn, using the same strategy as in (4.1) and (4.2) but for the terms like (see (1.2)):

$$\frac{z_i^{(1)}(t) - z_{i-1}^{(1)}(t)}{\| z_i^{(1)}(t) - z_{i-1}^{(1)}(t) \|_{R^2}} - \frac{z_i^{(2)}(t) - z_{i-1}^{(2)}(t)}{\| z_i^{(2)}(t) - z_{i-1}^{(2)}(t) \|_{R^2}}$$

(recall here (2.3)/(4.6)) and like

$$v_{i-1}(t) A \left((z_i^{(1)}(t) - z_{i-1}^{(1)}(t)) - (z_i^{(2)}(t) - z_{i-1}^{(2)}(t)) \right),$$

in place of $u_1 - u_2$ in (4.1)-(4.2), it is not difficult to show (in particular, making use of the inequality $|\| a \|_{R^2} - \| b \|_{R^2}| \leq \| a - b \|_{R^2}$) that, for our pair of solution and for sufficiently small T_0 (also satisfying (5.2)), there is a constant $M(T_0)$, nonincreasing as T_0 tends to 0+ such that

$$\| y_1 - y_2 \|_{(L^2(Q_T))^2} \leq M(T_0) \sqrt{T_0} \sum_{i=1}^{n} \| z_i^{(1)} - z_i^{(2)} \|_{C([0,T_0];R^2)} . \tag{5.3}$$

Combining (5.1)-(5.3) yields that $y_1 = y_2, z_i^{(1)} = z_i^{(2)}, i = 1, \ldots, n$ on some $[0, T_0]$, which provides the desirable uniqueness of solutions to (1.1)-(1.5), (2.1)-(2.3). This ends the proof of Theorem 2.1.

REFERENCES

[1] R. Cortez, The method of regilarized stokeslets, *SIAM. J. Sci. Comp.*, **23** (2001), pp. 1204-1225.

[2] L.J. Fauci and C.S. Peskin, A computational model of aquatic animal locomotion, *J. Comp. Physics*, **77** (1988), pp. 85-108.

[3] L.J. Fauci, Computational modeling of the swimming of biflagellated algal cells, *Contemporary Mathematics*, **141** (1993), pp. 91-102.

[4] A.Y. Khapalov, On bilinear controllability of the parabolic equation with the reaction-diffusion term satisfying Newton's Law, *Computational and Applied Mathematics, the special issue dedicated to the memory of J.-L. Lions*, **21** (2002), pp. 1-23.

[5] A.Y. Khapalov, Global non-negative controllability of the semilinear parabolic equation governed by bilinear control, *ESAIM: Contrôle, Optimisation et Calcul des Variations*, **7** (2002), pp. 269-283.

[6] A.Y. Khapalov, Controllability properties of a vibrating string with variable axial load, *Discrete and Cont. Dynamical Systems*, 11 (2004), pp. 311-324.

[7] O.H. Ladyzhenskaya, "The mathematical theory of viscous incompressible flow," Cordon and Breach, New-York, 1963.

[8] C.S. Peskin, Numerical analysis of blood flow in the heart, *J. Comp. Physics*, **25** (1977), pp. 220-252.

[9] C.S. Peskin and D.M. McQueen, A general method for the computer simulation of biological systems interacting with fluids, *SEB Symposium on biological fluid dynamics*, Leeds, England, July 5-8, 1994.

[10] K.R. Symon, *Mechanics*, Addison-Wesley series in Physics, 1971.

[11] R. Temam, *Navier-Stokes equations*, North-Holland, 1984.