MODEL OF HIV IN THE BLOOD

Alan Perelson and his colleagues* have analyzed a simple population dynamics model for viral infections such as HIV in the blood:

\[ \dot{T} = \lambda - dT - kTV, \]  \[ \dot{I} = -\delta I + kTV, \]  \[ \dot{V} = pI - cV. \]  

In these equations \( T \) is the concentration of healthy helper T cells, \( I \) is the concentration of infected helper T cells, and \( V \) is the concentration of virus particles in the blood. \( \lambda \) is a constant rate of production of T cells by the body, \( d \) is the death rate of healthy cells, \( \delta \) is the death rate of infected cells, \( p \) is the number of virus particles produced per infected cell and \( c \) is the rate that free virus particles leave the blood (due to both the immune system and new infections). \( k \) is the coefficient of a mass action term for the infection of a T cell by a virus particle. There are 6 parameters. The dependent variables and parameters are all non-negative. Dots denote differentiation with respect to time \( t \).

(a) This system has two equilibria. One is the healthy state: \( I = V = 0, \ T = \lambda / d \). Find the endemic equilibrium corresponding to an infection.

(b) Introduce the characteristic time \( t_0 = d^{-1} \) and the dimensionless time variable \( \tau = t/t_0 \). Obtain differential equations for the dynamics in terms of \( T', I' \) and \( V' \), where prime denotes differentiation with respect to \( \tau \).

(c) Define dimensionless variables \( X = T/A, Y = I/B \) and \( Z = V/C \) for suitable constants \( A, B, \) and \( C \) to obtain the equations in the form

\[ X' = 1 - X - XZ, \]  \[ Y' = -\alpha Y + XZ, \]  \[ Z' = \beta Y - \gamma Z \]  

Find expressions for \( A, B, \) and \( C \) in terms of the original model parameters. Note that there are now only three parameters: \( \alpha, \beta, \) and \( \gamma \). Find expressions for those too.

(d) Let \( x = X - 1, y = Y \) and \( z = Z \). Write down the differential equations for \( x', y' \) and \( z' \). What does the healthy steady state correspond to now?

*See, for example, Alan Perelson, Modelling viral and immune system dynamics, Nature Review | Immunology, Vol. 2., page 28, January 2002.