11.4 Tangent Planes & Linearization

Idea: \( z = f(x, y) \)
Now, normal vector for tan. plane is
\[ \langle 1, 0, f_x(a,b) \rangle \times \langle 0, 1, f_y(a,b) \rangle = \langle -f_x(a,b), -f_y(a,b), 1 \rangle \]

So, Eqn. of tan. plane is

\[ -f_x(a,b)(x-a) + f_y(a,b)(y-b) + 1(z-f(a,b)) = 0 \]

So:

\[ z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \]
See Thm 18, p. 644:

If $f_x(a,b)$ and $f_y(a,b)$ exist in a region surrounding $(a,b)$, and are continuous at $(a,b)$, then $f(x,y)$ is differentiable at $(a,b)$.
11.4 (4) \( f(x, y) = z = x \cdot e^{xy} \), \((2, 0, 2)\)

Tangent plane:
\[
z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)
\]

Here,
\[
f_x(x, y) = x \cdot e^{xy} + e^{xy}
\]
\[
f_y(x, y) = x \cdot e^{xy} \cdot x
\]
\[
f_x(2, 0) = 2(e^0 \cdot 0) + e^0 = 1
\]
\[
f_y(2, 0) = 2 \cdot e^0 \cdot 2 = 4
\]
\[
f(2, 0) = 2
\]
So: \[ z = 2 + 1(x-2) + 4(y-0) \]
\[ z = x + 4y \]

Note: for \((x,y)\) 'close' to \((2,0)\), we'll have \(f(x,y) = xe^{xy} \approx x + 4y\).

This new function is called the linearization of \(f\) at \((2,0)\).
So the formula for the linearization of \( f(x, y) \) at \( (x, y) = (a, b) \) is:

\[
L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b).
\]

11.4 (11) \( f(x, y) = 1 + x \ln(xy-5), \quad (2, 3) \).

\[
\frac{df}{dx} = x \left( \frac{1}{xy-5} \cdot y \right) + \ln(xy-5) \cdot 1
\]

\[
= \frac{xy}{xy-5} + \ln(xy-5).
\]

\[
\frac{df}{dy} = x \cdot \frac{1}{xy-5} \cdot x = \frac{x^2}{xy-5}.
\]
Both $f_x$ and $f_y$ are continuous on their domains, which are:

- $xy > 5$
- $xy \neq 5$

So both are continuous on $\{(x,y) | xy > 5\}$, and $(2,3)$ is in this region.

So by Thm 18, $f$ is differentiable at $(2,3)$.

To get $L(x,y)$:

- $f(2,3) = 1$
- $f_x(2,3) = 6 + 0 = 6$
- $f_y(2,3) = 4$
so \[ L(x,y) = 1 + 6(x-2) + 4(y-3) \]
\[ = 6x + 4y - 23. \]

( so for \((x,y)\) near \((2,3)\),

\[ 1 + x \ln(xy-5) \approx 6x + 4y - 23. \]

Note: for \(f(x,y,z,t)\), linearization at \((a,b,c,d)\) would be

\[ L(x,y,z,t) = f(a,b,c,d) + f_x(a,b,c,d)(x-a) + f_y(a,b,c,d)(y-b) \]
\[ + f_z(a,b,c,d)(z-c) \\
+ f_t(a,b,c,d)(t-d) \]
when input of $f$ goes from $(x,y)$ to $(x+\Delta x, y+\Delta y)$, change in $f(x,y)$ is:

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y).$$

Change in output of tangent plane is:

$$df = d\Delta z = f_x(x,y)\, dx + f_y(x,y)\, dy.$$