Double Integrals over Rectangular Regions

In set theory, \( A \cap B \triangleq \{ (x, y) \mid x \in A \text{ and } y \in B \} \). So, the rectangular region \( R = \{ (x, y) \mid a \leq x \leq b, c \leq y \leq d \} \subseteq \mathbb{R}^2 \) can be written as:

\[
[a, b] \times [c, d].
\]

Recall that to find the volume of a solid, you can integrate the cross-sectional area perpendicular to some axis:

\[
\text{Volume} = \int_a^b A(x) \, dx.
\]

You can find volume which lies under \( z = f(x, y) \) and above \( [a, b] \times [c, d] \) in 2 ways:

1) \( z = f(x, y) \)

\[
A(x) = \int_c^d f(x, y) \, dy.
\]

\[
\text{Volume} = \int_a^b A(x) \, dx = \int_a^b \left( \int_c^d f(x, y) \, dy \right) \, dx.
\]

2) \( z = f(x, y) \)

\[
A(y) = \int_a^b f(x, y) \, dx.
\]

\[
\text{Volume} = \int_c^d A(y) \, dy = \int_c^d \left( \int_a^b f(x, y) \, dx \right) \, dy.
\]
Double Integral can also be defined in terms of Riemann sums:

\[ \Delta x = \frac{b-a}{m}, \quad \Delta y = \frac{d-c}{n} \]

\[ \Delta A = \text{area of each sub-rectangle} = \Delta x \Delta y \]

Point \((x_i, y_j)\) is "for" corner of sub-rectangle \(A_{ij}\)

Volume of column which lies above \(A_{ij}\) is approximately

\[ \Delta A \cdot f(x_i, y_j) \]

So, total volume \( \approx \sum_{i=1}^{m} \left( \sum_{j=1}^{n} f(x_i, y_j) \Delta A \right) = \sum_{j=1}^{n} \left( \sum_{i=1}^{m} f(x_i, y_j) \Delta A \right) \)

When we let \(n \to \infty\) and \(m \to \infty\), we get

Volume \( = \lim_{m \to \infty} \left( \lim_{n \to \infty} \left( \sum_{i=1}^{m} \left( \sum_{j=1}^{n} f(x_i, y_j) \Delta A \right) \right) \right) \)

\( = \lim_{m \to \infty} \left( \sum_{i=1}^{m} \left( \lim_{n \to \infty} \left( \sum_{j=1}^{n} f(x_i, y_j) \Delta y \right) \right) \Delta x \right) \)

\( = \lim_{m \to \infty} \left( \sum_{i=1}^{m} \left( \int_{c}^{d} f(x_i, y) \, dy \right) \Delta x \right) \)

\( = \int_{a}^{b} \left( \int_{c}^{d} f(x_i, y) \, dy \right) \, dx \)

or, we could exchange the order of the sums and limits, and get

\( \int_{c}^{d} \left( \int_{a}^{b} f(x_i, y) \, dx \right) \, dy \).