Last time: For \( \sum_{k=1}^{\infty} \frac{3^k}{k^2} (x+4)^k \),

Center: \( a = -4 \)

Radius of convergence? Use Ratio Test:

\[
\lim_{k \to \infty} \left| \frac{(k+1)^{th \text{ term}}}{k^{th \text{ term}}} \right| = \lim_{k \to \infty} \left| \frac{\frac{3^{k+1}(x+4)^{k+1}}{(k+1)^2}}{\frac{3^k(x+4)^k}{k^2}} \right|
\]

\[
= \lim_{k \to \infty} \left| \frac{3(x+4)k^2}{(k+1)^2} \right| = \lim_{k \to \infty} \left( \frac{3k^2}{(k+1)^2} / |x+4| \right)
\]

\[
= |x+4| \lim_{k \to \infty} \left( \frac{3k^2}{(k+1)^2} \right) = |x+4| \cdot 3 = 3|x+4|
\]
So need $3|x+4| < 1$ for convergence

$\implies |x+4| < \frac{1}{3}$

ROC is $\frac{1}{3}$.

Generically: $|x-a| < k$:

\[
\frac{a-k}{a+k} < x < \frac{a+k}{a-k}
\]

Interval of Convergence?

$|x+4| < \frac{1}{3} \iff -\frac{1}{3} < x+4 < \frac{1}{3}$

$\iff -\frac{13}{3} < x < -\frac{11}{3}$

End points $A$ and $B$.

When $x = -\frac{13}{3}$, series becomes

$$\sum_{k=1}^{\infty} \frac{(-\frac{3}{3})^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

Converges by A.S.T.

Since $\frac{1}{k^2} \to 0$. 

$$A^k B^k = (AB)^k$$
When \( x = \frac{-11}{3} \), series becomes
\[
\sum_{k=1}^{\infty} \frac{3^k}{k^2} \left( \frac{1}{3} \right)^k = \sum_{k=1}^{\infty} \frac{1}{k^2},
\]
convergent p-series, \( p = 2 > 1 \).

So \( T = \left[ \frac{13}{3}, \frac{11}{3} \right] \).

Like #4: Consider curve \( \begin{cases} x = 4t^2 - 1 \\ y = \sin(\pi t) \end{cases} \)

a) What \( t \)-value corresponds to the point \((x, y) = (0, -1)\) on this curve?

Sol: \( 4t^2 - 1 = 0 \) \( \Rightarrow t^2 = \frac{1}{4} \) \( \Rightarrow t = \pm \frac{1}{2} \).

\( y = \sin(\pi t) = -1 \) \( \Rightarrow \) only \( t = -\frac{1}{2} \) works.

So \( t = -\frac{1}{2} \).
b) Find the equation of the tangent line to this curve at (0, -1).

\[ y - (-1) = m(x - 0) \]

To get \( m \):

\[ \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\pi \cos(\pi t)}{8t^2} \]

So,

\[ m = \frac{\pi \cos(\pi t)}{8t^2} \bigg|_{t=-\frac{1}{2}} = \frac{\pi \cos(-\frac{\pi}{2})}{8(-\frac{1}{2})} = 0. \]

So the line is \( y + 1 = 0(x - 0) \), or \( y = -1 \).
Question 4. Consider the curve defined by the parametric equations

\[ x = \sin t - t \]
\[ y = 2 - 2 \cos t \]

A. (5 points) What value of \( t \) corresponds to the point \((\pi, 4)\) on the graph above?

B. (5 points) As \( t \) increases, is the curve traced from left to right or from right to left?

\[ a + t = 0, \quad x = 0, \quad y = 0 \]
\[ a + t = \frac{\pi}{2}, \quad x = 1 - \frac{\pi}{2} \approx -0.5, \quad y = 2 \]

C. (10 points) Find an expression in terms of \( t \) for \( \frac{dy}{dx} \), the slope of the tangent line to the curve above.
To parameterize a line, like \( y = 3x + 2 \), just do:

\[
\begin{align*}
  x &= t \\
  y &= 3t + 2
\end{align*}
\]

To parameterize a circle, centered at \((0,0)\), with radius 5:

\[
\begin{align*}
  x &= 5 \cos(t) \\
  y &= 5 \sin(t)
\end{align*}
\]
Question 5. The polar curve $r = 1 + 2 \cos 2\theta$ is depicted.

A. (5 points) Give any value of $\theta$ that produces the origin as a point on this curve; this occurs when $r = 0$.

\[ 1 + 2 \cos(2\theta) = 0 \]
\[ \cos(2\theta) = -\frac{1}{2} \]

B. (5 points) Give any value of $\theta$ that corresponds to a point on this curve where the tangent line is horizontal (you may do this by inspection; you are not required to prove your answer formally using part C.)

C. (10 points) Compute an expression for $\frac{dy}{dx}$, the slope of the tangent line to this curve, in terms of $\theta$.

\[ x = r \cos(\theta) = (1 + 2 \cos(2\theta)) \cos(\theta) \]
\[ y = r \sin(\theta) = (1 + 2 \cos(2\theta)) \sin(\theta) \]

\[ \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\text{use product rule}}{\text{use product rule}} \]
Question 6. (10 points) The polar curve \( r = \sqrt{3 + \cos \theta} \), \( 0 \leq \theta \leq 2\pi \) is depicted. Find the area of the region bounded by this curve (which is not a circle.)

\[
A = \int_{0}^{\pi} \frac{1}{2} (\sqrt{3 + \cos \theta})^2 d\theta
\]

\[
= \int_{0}^{\pi} \frac{1}{2} (3 + \cos \theta) d\theta
\]

\[
= \left[ \frac{3\theta}{2} + \sin \theta \right]_{0}^{\pi}
\]

\[
= \frac{3\pi}{2} + \sin \pi - (\frac{3\cdot0}{2} + \sin 0)
\]

\[
= \frac{3\pi}{2} + 0 - 0
\]

\[
= \frac{3\pi}{2}
\]

\[
\text{Area} = \int_{a}^{b} \frac{1}{2} r^2 d\theta = \int_{a}^{b} \frac{1}{2} (f(\theta))^2 d\theta
\]

\[
r = f(\theta)
\]
\(\theta = a, b\)
\[ \theta = \frac{3\pi}{2} \]

\[ z = 0 \]

\[ y = 0 \]

Graph \( V = z + 2 \sin(\theta) \) in \( xy \) plane.
Question 7. (5 points each) Let $\mathbf{u} = (1,3)$ and $\mathbf{v} = (-4,1)$.

A. Illustrate using the parallelogram rule to show $\mathbf{u} + \mathbf{v} = (-3,4)$ on the axes below.

\[ \mathbf{u} \cdot \mathbf{v} = (1)(-4) + (3)(1) = -1. \]

\[ \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos(\theta) \]

B. Compute $\mathbf{u} \cdot \mathbf{v} = (1)(-4) + (3)(1) = -1$.

C. Determine whether the angle between $\mathbf{u}$ and $\mathbf{v}$ is acute, right, or obtuse and justify your choice using your answer from part B.

$\theta$ must be $> 90^\circ$.

D. Compute $|\mathbf{u}| = \sqrt{1^2 + 3^2} = \sqrt{10}$.

$\mathbf{v} = (-4,1)$.
D. Find $\mathbf{n} \times \mathbf{v}$.

C. Find the projection of $\mathbf{n}$ onto $\mathbf{v}$.

B. Find the exact angle $\theta$, $0 \leq \theta \leq \pi$, between $\mathbf{n}$ and $\mathbf{v}$; it is "nice" multiple of $\pi$.

A. Find $\mathbf{n} \cdot \mathbf{v}$.

Question 8: (5 points each) Let $\mathbf{a} = \langle 1, 2 \rangle$ and $\mathbf{b} = \langle 1, 0 \rangle$.

$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 2 = 2$.
Question 9. (10 points) Find an equation of the line passing through the two points (1,3,4) and (5, -1, 0). Write your answer in vector form.

\[ \langle x, y, z \rangle = \langle 1, 3, 4 \rangle + \langle 4, -4, -4 \rangle t \]

\[ \vec{v}(t) = \langle 1 + 4t, 3 - 4t, 4 - 4t \rangle \]