more 9.2

A large part of 9.2 involves finding power series representations for functions which can be related to \( \frac{1}{1-x} \).

We know

\[
\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \ldots = \frac{1}{1-x},
\]

for \(-1 < x < 1\).

It follows that

\[
\sum_{k=0}^{\infty} (\alpha)^k = \frac{1}{1-(\alpha)}, \text{ for } -1 < \alpha < 1.
\]
Examples: Find a power series representation for \( f(x) = \frac{2x^2}{3+x^2} \).

SOL: \( f(x) = 2x^2 \cdot \frac{1}{3(1+x^2/3)} = \frac{2x^2}{3} \cdot \frac{1}{1-(-x^2/3)} = \frac{2x^2}{3} \sum_{k=0}^{\infty} \left( \frac{-x^2}{3} \right)^k \).

\[
\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{3^k} = \sum_{k=0}^{\infty} \frac{2(-1)^k x^{2k+2}}{3^{k+1}}
\]

For C: \( -1 < \frac{-x^2}{3} < 1 \) \( \Rightarrow \) \( 1 > \frac{x^2}{3} > -1 \)

\( \Rightarrow 3 > x^2 > -3 \) \( \Rightarrow \) \( 0 < x^2 < 3 \)

\( \Rightarrow |x| < \sqrt{3} \) \( \Rightarrow -\sqrt{3} < x < \sqrt{3} \)
Another: Find power series representation for

\[ f(x) = \tan^{-1}(x), \text{ and its I.O.S.} \]

\[ \text{Sol:} \quad \frac{d}{dx} \left( \tan^{-1}(x) \right) = \frac{1}{1 + x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} \]

and \[ \frac{1}{1 + x^2} = \frac{1}{1 - (-x^2)} = \sum_{k=0}^{\infty} (-1)^k x^{2k} \]

So \[ \tan^{-1}(x) = \int \left( \sum_{k=0}^{\infty} (-1)^k x^{2k} \right) \, dx \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C. \]
Shouldn't have +C, so:

above it have

\[ \tan^{-1}(x) = \left( \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right) + C. \]

\[ \text{evaluate at } x=0: \]

\[ \tan^{-1}(0) = (0 + 0 + \ldots) + C \]

\[ \Rightarrow 0 = C. \]

So, \( \tan^{-1}(x) = \sum_{k=0}^{\infty} \frac{(1)^k x^{2k+1}}{(2k+1)} \). \quad \text{valid for } -1 \leq x \leq 1 \text{, at least}.

\[ \text{T.O.C: know converges for } -1 < x < 1, \]

may or may not converge when \( x = 1, x = -1 \).
When $x = 1$, series is \[ \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \] a convergent alternating series.

When $x = -1$, series is \[ \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cdot (-1)^{2k+1} \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cdot (-1)^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{2k+1} \]

So I.O.C. for the series I've found for $f_n^{-1}(x)$ is $-1 \leq x \leq 1$.

$$e^{\pi i} 2^k = (e^{-i\pi})^{2^k} = (1)^{2^k} = 1.$$
Another. Find series rep. centered at \( x = 2 \)

for \( f(x) = \frac{3}{9-x} \)

\[
\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k
\]

\[
f(x) = 3 \cdot \frac{1}{2 - (x-2)} = \frac{3}{2} \cdot \frac{1}{1 - \left| \frac{1}{2} (x-2) \right|}
\]

\[
\Rightarrow -2 < (x-2) < 2 \\
0 < x < 4.
\]
9.3 Taylor Series Formula.

Can find a power series representation, centered at any $a$, for any $f(x)$, which has all orders of derivatives at $x = a$, as follows:

\[
    f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k
\]

\[
    = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots
\]
In the above eqn,

set \( x = a \), get \( f'(a) = c_0 \)

take deriv,
then set \( x = a \):

\[ f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots \]

\[ f'(a) = c_1 \]

take deriv, again,
set \( x = a \):

\[ f''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + 4 \cdot 3c_4(x-a)^2 + \ldots \]

\[ f''(a) = 2c_2 \Rightarrow c_2 = \frac{1}{2} f''(a) \]

take deriv again,
set \( x = a \):

\[ f'''(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + 5 \cdot 4 \cdot 3c_5(x-a)^2 + \ldots \]

\[ f'''(a) = 3 \cdot 2c_3 \Rightarrow c_3 = \frac{1}{3 \cdot 2} f'''(a) \]

\[ \vdots \]
\[ C_n = \frac{1}{n!} f^{(n)}(a) \]

This tells us what every coefficient in the series must be.

we'll use this on Friday.